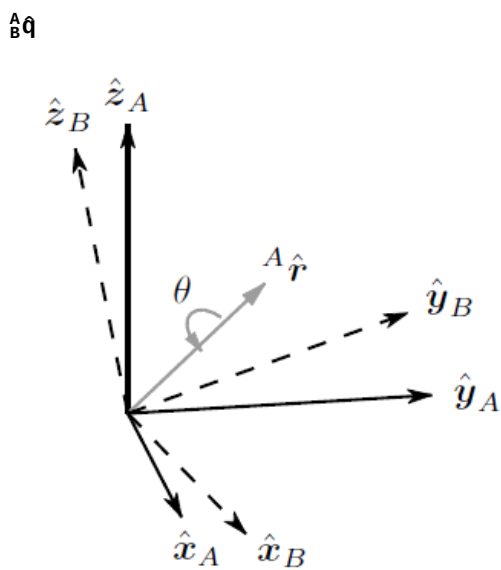


An efficient orientation filter for inertial and inertial/magnetic sensor arrays

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Quaternion representation

- A quaternion is a four-dimensional complex number
- An arbitrary orientation of frame B relative to frame A



- A rotation of angle θ around an axis ${}^A\hat{r}$ defined in frame A

$${}^A\hat{r} = (r_x \quad r_y \quad r_z)$$

- normalized four-dimensional vector

$${}^A\hat{q} = (q_1 \quad q_2 \quad q_3 \quad q_4) = (\cos[\frac{\theta}{2}] \quad -r_x \sin[\frac{\theta}{2}] \quad -r_y \sin[\frac{\theta}{2}] \quad -r_z \sin[\frac{\theta}{2}])$$

- the quaternion paramentrization obeys

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

- hypercomplex numbers {i,j,k}

$$\begin{aligned} &a + b\,i + c\,j + d\,k \\ &i^2 = j^2 = k^2 = -1 \\ &i\,j = -j\,i = k \\ &j\,k = -k\,j = i \\ &k\,i = -i\,k = j \end{aligned}$$

- A quaternion conjugate

$${}^A\hat{q}^* = {}^B\hat{q} = (q_1 \quad -q_2 \quad -q_3 \quad -q_4)$$

- A quaternion product $\otimes$

$${}^A\hat{q} = {}^B\hat{q} \otimes {}^A\hat{q}$$

■ Hamilton rule

$$\begin{aligned}\mathbf{a} \otimes \mathbf{b} &= (\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4) \otimes (\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4) \\ &= (\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4) \cdot \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \\ -\mathbf{b}_2 & \mathbf{b}_1 & -\mathbf{b}_4 & \mathbf{b}_3 \\ -\mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_1 & -\mathbf{b}_2 \\ -\mathbf{b}_4 & -\mathbf{b}_3 & \mathbf{b}_2 & \mathbf{b}_1 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{a}_1 \mathbf{b}_1 - \mathbf{a}_2 \mathbf{b}_2 - \mathbf{a}_3 \mathbf{b}_3 - \mathbf{a}_4 \mathbf{b}_4 \\ \mathbf{a}_1 \mathbf{b}_2 + \mathbf{a}_2 \mathbf{b}_1 + \mathbf{a}_3 \mathbf{b}_4 - \mathbf{a}_4 \mathbf{b}_3 \\ \mathbf{a}_1 \mathbf{b}_3 - \mathbf{a}_2 \mathbf{b}_4 + \mathbf{a}_3 \mathbf{b}_1 + \mathbf{a}_4 \mathbf{b}_2 \\ \mathbf{a}_1 \mathbf{b}_4 + \mathbf{a}_2 \mathbf{b}_3 - \mathbf{a}_3 \mathbf{b}_2 + \mathbf{a}_4 \mathbf{b}_1 \end{pmatrix}_{\text{T}}\end{aligned}$$

■ Euler axis

$$\hat{\mathbf{e}} = \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix}$$

■ Angle

$$\theta$$

■ Vector's Elements

$$\begin{aligned}\mathbf{q}_1 &= \text{Cos}\left[\frac{\theta}{2}\right] \\ \mathbf{q}_2 &= \mathbf{e}_x \text{Sin}\left[\frac{\theta}{2}\right] \\ \mathbf{q}_3 &= \mathbf{e}_y \text{Sin}\left[\frac{\theta}{2}\right] \\ \mathbf{q}_4 &= \mathbf{e}_z \text{Sin}\left[\frac{\theta}{2}\right]\end{aligned}$$

■ Half Angle Identities

$$\text{Sin}\left[\frac{\theta}{2}\right] = \pm \sqrt{\frac{1 - \text{Cos}[\theta]}{2}}$$

$$\text{Cos}\left[\frac{\theta}{2}\right] = \pm \sqrt{\frac{1 + \text{Cos}[\theta]}{2}}$$

Choice of $\pm$ sign depends on the quadrant in which $\frac{\theta}{2}$ lies.

■ A three dimensional vector

■ Quaternion ${}^A\hat{\mathbf{q}}$

$$\begin{aligned}\mathbf{B}_{\mathcal{U}} &= {}^A\hat{\mathbf{q}} \otimes \mathbf{A}_{\mathcal{U}} \otimes {}^A\hat{\mathbf{q}}^* \\ \mathbf{C}_{\mathcal{U}} &= {}^B\hat{\mathbf{q}} \otimes \mathbf{B}_{\mathcal{U}} \otimes {}^B\hat{\mathbf{q}}^* \\ \mathbf{C}_{\mathcal{U}} &= {}^B\hat{\mathbf{q}} \otimes \mathbf{B}_{\mathcal{U}} \otimes {}^B\hat{\mathbf{q}}^* = {}^B\hat{\mathbf{q}} \otimes ({}^A\hat{\mathbf{q}} \otimes \mathbf{A}_{\mathcal{U}} \otimes {}^A\hat{\mathbf{q}}^*) \otimes {}^B\hat{\mathbf{q}}^* \\ {}^A\hat{\mathbf{q}} &= {}^B\hat{\mathbf{q}} \otimes {}^A\hat{\mathbf{q}}\end{aligned}$$

■ Rotation Matrix ${}^B\mathbf{R}_A$

$$\begin{aligned}\mathbf{B}_{\mathcal{U}} &= {}^B\mathbf{R}_A \cdot \mathbf{A}_{\mathcal{U}} \\ \mathbf{C}_{\mathcal{U}} &= {}^B\mathbf{R}_B \cdot \mathbf{B}_{\mathcal{U}} \\ \mathbf{C}_{\mathcal{U}} &= {}^C\mathbf{R}_A \cdot \mathbf{A}_{\mathcal{U}} = {}^C\mathbf{R}_B \cdot {}^B\mathbf{R}_A \cdot \mathbf{A}_{\mathcal{U}} = {}^C\mathbf{R}_B \cdot \mathbf{B}_{\mathcal{U}} \\ {}^C\mathbf{R}_A &= {}^C\mathbf{R}_B \cdot {}^B\mathbf{R}_A \\ {}^E\mathbf{R}_A &= {}^E\mathbf{R}_V \cdot {}^V\mathbf{R}_A \\ {}^E\mathbf{R}_B &= {}^E\mathbf{R}_V \cdot {}^V\mathbf{R}_B \\ {}^E\mathbf{R}_B &= {}^E\mathbf{R}_V \cdot {}^V\mathbf{R}_A \cdot {}^A\mathbf{R}_B = {}^E\mathbf{R}_V \cdot {}^V\mathbf{R}_B = {}^E\mathbf{R}_A \cdot {}^A\mathbf{R}_B\end{aligned}$$

■ Euler Angle (Roll $\phi \hat{x}_B$, Pitch $\theta \hat{y}_B$, Yaw $\psi \hat{z}_B$, ZYX) $\rightarrow$ Rotation Matrix ${}^B\mathbf{R}_A$

$$\begin{aligned}{}^B\mathbf{R}_A &= \begin{pmatrix} \mathbf{r}_{11} & \mathbf{r}_{12} & \mathbf{r}_{13} \\ \mathbf{r}_{21} & \mathbf{r}_{22} & \mathbf{r}_{23} \\ \mathbf{r}_{31} & \mathbf{r}_{32} & \mathbf{r}_{33} \end{pmatrix} = \mathbf{R}_z[\psi] \mathbf{R}_y[\theta] \mathbf{R}_x[\phi] \\ &= \begin{pmatrix} \text{Cos}[\psi] & -\text{Sin}[\psi] & 0 \\ \text{Sin}[\psi] & \text{Cos}[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}[\theta] & 0 & \text{Sin}[\theta] \\ 0 & 1 & 0 \\ -\text{Sin}[\theta] & 0 & \text{Cos}[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[\phi] & -\text{Sin}[\phi] \\ 0 & \text{Sin}[\phi] & \text{Cos}[\phi] \end{pmatrix} \\ &= \begin{pmatrix} \text{Cos}[\psi] \text{Cos}[\theta] & \text{Cos}[\psi] \text{Sin}[\theta] \text{Sin}[\phi] - \text{Sin}[\psi] \text{Cos}[\phi] & \text{Cos}[\psi] \text{Sin}[\theta] \text{Cos}[\phi] + \text{Sin}[\psi] \text{Sin}[\phi] \\ \text{Sin}[\psi] \text{Cos}[\theta] & \text{Sin}[\psi] \text{Sin}[\theta] \text{Sin}[\phi] + \text{Cos}[\psi] \text{Cos}[\phi] & \text{Sin}[\psi] \text{Sin}[\theta] \text{Cos}[\phi] - \text{Cos}[\psi] \text{Sin}[\phi] \\ -\text{Sin}[\theta] & \text{Cos}[\theta] \text{Sin}[\phi] & \text{Cos}[\theta] \text{Cos}[\phi] \end{pmatrix}\end{aligned}$$

■ Axis / Angle → Rotation Matrix ${}^B_A\mathbf{R}$

$$\begin{aligned} \mathbf{R}[\gamma, \mathbf{r}] &= \mathbf{R}_z[\alpha] \cdot \mathbf{R}_y[\beta] \cdot \mathbf{R}_z[\gamma] \cdot \mathbf{R}_y[-\beta] \cdot \mathbf{R}_z[-\alpha] \\ &= \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[\gamma] & -\sin[\gamma] & 0 \\ \sin[\gamma] & \cos[\gamma] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[-\beta] & 0 & \sin[-\beta] \\ 0 & 1 & 0 \\ -\sin[-\beta] & 0 & \cos[-\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[-\alpha] & -\sin[-\alpha] & 0 \\ \sin[-\alpha] & \cos[-\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ {}^B_A\mathbf{R} &= \begin{pmatrix} r_x^2 (1 - \cos[\gamma]) + \cos[\gamma] & r_x r_y (1 - \cos[\gamma]) - r_z \sin[\gamma] & r_x r_z (1 - \cos[\gamma]) + r_y \sin[\gamma] \\ r_x r_y (1 - \cos[\gamma]) + r_z \sin[\gamma] & r_y^2 (1 - \cos[\gamma]) + \cos[\gamma] & r_y r_z (1 - \cos[\gamma]) - r_x \sin[\gamma] \\ r_x r_z (1 - \cos[\gamma]) - r_y \sin[\gamma] & r_y r_z (1 - \cos[\gamma]) + r_x \sin[\gamma] & r_z^2 (1 - \cos[\gamma]) + \cos[\gamma] \end{pmatrix} \\ \cos\left[\frac{\gamma}{2}\right] &= \pm \sqrt{\frac{1 + \cos[\gamma]}{2}} \quad \sin\left[\frac{\gamma}{2}\right] = \pm \sqrt{\frac{1 - \cos[\gamma]}{2}} \end{aligned}$$

■ Quaternion ${}^B_A\hat{\mathbf{q}}$ → Rotation Matrix ${}^B_A\mathbf{R}$

$${}^B_A\hat{\mathbf{q}} = (q_1 \quad q_2 \quad q_3 \quad q_4) = \left(\cos\left[\frac{\theta}{2}\right] \quad -r_x \sin\left[\frac{\theta}{2}\right] \quad -r_y \sin\left[\frac{\theta}{2}\right] \quad -r_z \sin\left[\frac{\theta}{2}\right] \right)$$

$${}^B_A\mathbf{R} = \begin{pmatrix} 2q_1^2 - 1 + 2q_2^2 & 2(q_2q_3 + q_1q_4) & 2(q_2q_4 - q_1q_3) \\ 2(q_2q_3 - q_1q_4) & 2q_1^2 - 1 + 2q_3^2 & 2(q_3q_4 + q_1q_2) \\ 2(q_2q_4 + q_1q_3) & 2(q_3q_4 - q_1q_2) & 2q_1^2 - 1 + 2q_4^2 \end{pmatrix}$$

■ Quaternion ${}^B_A\hat{\mathbf{q}}$ → Rotation Matrix ${}^B_A\mathbf{R}$ → Euler Angle (Roll $\phi \hat{x}_B$, Pitch $\theta \hat{y}_B$, Yaw $\psi \hat{z}_B$, ZYX)

ϕ around $\hat{x}_B$, θ around $\hat{y}_B$, ψ around $\hat{z}_B$

$$\phi = \text{ArcTan}\left[{}^B_A\mathbf{R}_{33}, {}^B_A\mathbf{R}_{32}\right] = \text{ArcTan}\left[2q_1^2 - 1 + 2q_4^2, 2(q_3q_4 - q_1q_2)\right]$$

$$\theta = -\text{ArcSin}\left[{}^B_A\mathbf{R}_{31}\right] = -\text{ArcTan}\left[\sqrt{1 - {}^B_A\mathbf{R}_{31}^2}, {}^B_A\mathbf{R}_{31}\right] = -\text{ArcSin}\left[2(q_2q_4 + q_1q_3)\right]$$

$$\psi = \text{ArcTan}\left[{}^B_A\mathbf{R}_{11}, {}^B_A\mathbf{R}_{21}\right] = \text{ArcTan}\left[2q_1^2 - 1 + 2q_2^2, 2(q_2q_3 - q_1q_4)\right]$$

$$\text{Roll} = \text{atan2}\left(\mathbf{R}_{32}, \mathbf{R}_{33}\right)$$

$$\text{Pitch} = -\text{asin}\left(\mathbf{R}_{31}\right)$$

$$\text{Yaw} = \text{atan2}\left(\mathbf{R}_{21}, \mathbf{R}_{11}\right)$$

■ Rotation Matrix ${}^B_A\mathbf{R}$ → Quaternion ${}^B_A\hat{\mathbf{q}}$

$$\begin{aligned} q_1 &= \frac{1}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1} \\ q_2 &= -\frac{\text{Sign}[r_{32} - r_{23}]}{2} \sqrt{r_{11} - r_{22} - r_{33} + 1} \\ q_3 &= -\frac{\text{Sign}[r_{13} - r_{31}]}{2} \sqrt{r_{22} - r_{33} - r_{11} + 1} \\ q_4 &= -\frac{\text{Sign}[r_{21} - r_{12}]}{2} \sqrt{r_{33} - r_{11} - r_{22} + 1} \end{aligned}$$

■ Euler Angle (Roll $\phi \hat{x}_B$, Pitch $\theta \hat{y}_B$, Yaw $\psi \hat{z}_B$, ZYX) → Quaternion ${}^B_A\hat{\mathbf{q}}$

$$\begin{aligned} q_1 &= \frac{1}{2} \sqrt{1 + \cos[\theta] \cos[\phi] + \cos[\theta] \cos[\psi] + \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi]} \\ q_2 &= -\frac{1}{2} \frac{\text{Sign}[\cos[\theta] \sin[\phi] + \cos[\psi] \sin[\phi] - \cos[\phi] \sin[\theta] \sin[\psi]]}{\sqrt{1 - \cos[\theta] \cos[\phi] + \cos[\theta] \cos[\psi] - \cos[\phi] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi]}} \\ q_3 &= -\frac{1}{2} \frac{\text{Sign}[\sin[\theta] + \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi]]}{\sqrt{1 - \cos[\theta] \cos[\phi] - \cos[\theta] \cos[\psi] + \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi]}} \\ q_4 &= -\frac{1}{2} \frac{\text{Sign}[-\cos[\psi] \sin[\theta] \sin[\phi] + \cos[\theta] \sin[\psi] + \cos[\phi] \sin[\psi]]}{\sqrt{1 + \cos[\theta] \cos[\phi] - \cos[\theta] \cos[\psi] - \cos[\phi] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi]}} \end{aligned}$$

$$\mathbf{Q}[\phi, \theta, \psi] = \left\{ \frac{1}{2} \sqrt{2 + 2\cos[\phi]}, -\frac{1}{2} \sqrt{2 - 2\cos[\phi]} \text{Sign}[\sin[\phi]], \theta, \psi \right\} = \left\{ \cos\left[\frac{\phi}{2}\right], -\sin\left[\frac{\phi}{2}\right], \theta, \psi \right\}$$

$$\mathbf{Q}[\theta, \phi, \psi] = \left\{ \frac{1}{2} \sqrt{2 + 2\cos[\theta]}, \theta, -\frac{1}{2} \sqrt{2 - 2\cos[\theta]} \text{Sign}[\sin[\theta]], \psi \right\} = \left\{ \cos\left[\frac{\theta}{2}\right], \theta, -\sin\left[\frac{\theta}{2}\right], \psi \right\}$$

$$\mathbf{Q}[\theta, \phi, \psi] = \left\{ \frac{1}{2} \sqrt{2 + 2\cos[\psi]}, \theta, \psi, -\frac{1}{2} \sqrt{2 - 2\cos[\psi]} \text{Sign}[\sin[\psi]] \right\} = \left\{ \cos\left[\frac{\psi}{2}\right], \theta, \psi, -\sin\left[\frac{\psi}{2}\right] \right\}$$

$$\left\{ \cos\left[\frac{\phi}{2}\right], -\sin\left[\frac{\phi}{2}\right], \theta, \psi \right\} \otimes \left\{ \cos\left[\frac{\theta}{2}\right], \theta, -\sin\left[\frac{\theta}{2}\right], \psi \right\} \otimes \left\{ \cos\left[\frac{\psi}{2}\right], \theta, \psi, -\sin\left[\frac{\psi}{2}\right] \right\}$$

$$\begin{aligned} q_1 &= \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\psi}{2}\right] + \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ q_2 &= -\cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] \sin\left[\frac{\phi}{2}\right] + \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ q_3 &= -\cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\psi}{2}\right] \sin\left[\frac{\theta}{2}\right] - \cos\left[\frac{\theta}{2}\right] \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ q_4 &= \cos\left[\frac{\psi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\phi}{2}\right] - \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\psi}{2}\right] \end{aligned}$$

■ Jacobian matrix and determinant

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m, \text{ (} m \times n \text{ matrix)}$

$x \in \mathbb{R}^n, f[x] \in \mathbb{R}^m$

$J_{i,j} = \frac{\partial f_i}{\partial x_j}$

$\nabla f = J^T f$

$J = \frac{df}{dx} = \left(\frac{\partial f}{\partial x_1} \quad \dots \quad \frac{\partial f}{\partial x_n} \right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \partial_{x_1} f_1 & \dots & \partial_{x_n} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_m & \dots & \partial_{x_n} f_m \end{pmatrix}$

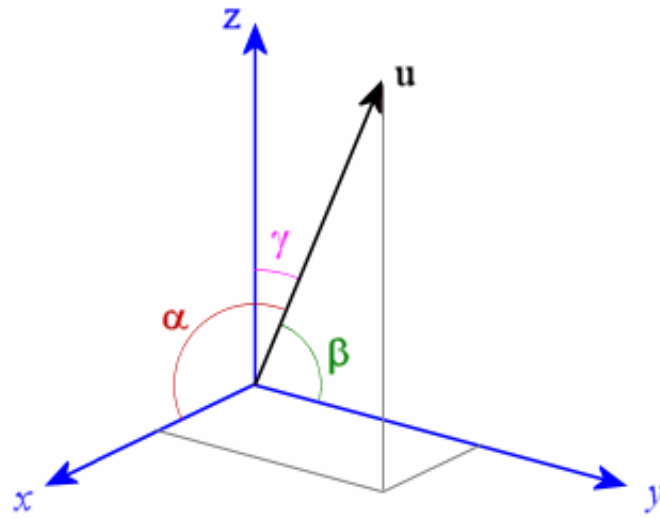
Orientation

■ Inner Product Space

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = |\hat{\mathbf{x}}| |\hat{\mathbf{y}}| \cos[\theta]$$

$$\cos[\theta] = \frac{\hat{\mathbf{x}} \cdot \hat{\mathbf{y}}}{|\hat{\mathbf{x}}| |\hat{\mathbf{y}}|}$$

■ Representation of orientation



$$\hat{\mathbf{u}} = u_1 \hat{\mathbf{e}}_1 + u_2 \hat{\mathbf{e}}_2 + u_3 \hat{\mathbf{e}}_3$$

$$a^2 + b^2 + c^2 = \cos^2[\alpha] + \cos^2[\beta] + \cos^2[\gamma] = 1$$

$$\hat{\mathbf{v}} = \{v_x, v_y, v_z\} \cdot \{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\} = v_x \hat{\mathbf{e}}_x + v_y \hat{\mathbf{e}}_y + v_z \hat{\mathbf{e}}_z$$

$$|\hat{\mathbf{v}}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$a = \cos[\alpha] = \frac{\hat{\mathbf{v}} \cdot \hat{\mathbf{e}}_x}{|\hat{\mathbf{v}}|} = \frac{(v_x \hat{\mathbf{e}}_x + v_y \hat{\mathbf{e}}_y + v_z \hat{\mathbf{e}}_z) \cdot \hat{\mathbf{e}}_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_x}{|\hat{\mathbf{v}}|}$$

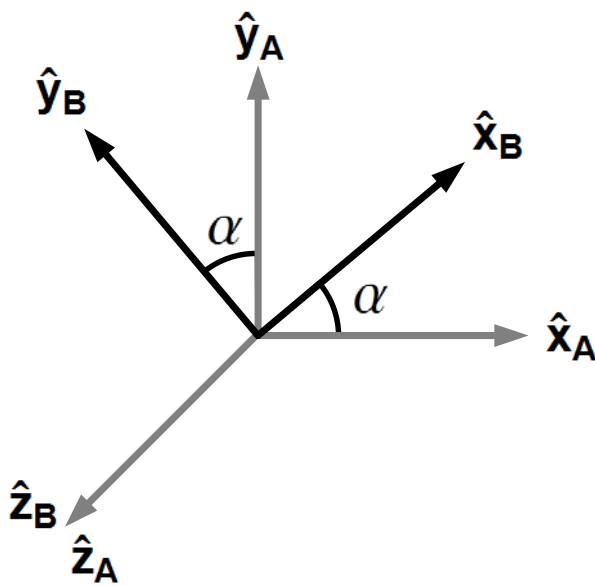
$$b = \cos[\beta] = \frac{\hat{\mathbf{v}} \cdot \hat{\mathbf{e}}_y}{|\hat{\mathbf{v}}|} = \frac{(v_x \hat{\mathbf{e}}_x + v_y \hat{\mathbf{e}}_y + v_z \hat{\mathbf{e}}_z) \cdot \hat{\mathbf{e}}_y}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_y}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_y}{|\hat{\mathbf{v}}|}$$

$$c = \cos[\gamma] = \frac{\hat{\mathbf{v}} \cdot \hat{\mathbf{e}}_z}{|\hat{\mathbf{v}}|} = \frac{(v_x \hat{\mathbf{e}}_x + v_y \hat{\mathbf{e}}_y + v_z \hat{\mathbf{e}}_z) \cdot \hat{\mathbf{e}}_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_z}{|\hat{\mathbf{v}}|}$$

$$\begin{aligned} \hat{\mathbf{u}} &= \frac{\hat{\mathbf{v}}}{|\hat{\mathbf{v}}|} = \frac{v_x}{|\hat{\mathbf{v}}|} \hat{\mathbf{e}}_x + \frac{v_y}{|\hat{\mathbf{v}}|} \hat{\mathbf{e}}_y + \frac{v_z}{|\hat{\mathbf{v}}|} \hat{\mathbf{e}}_z \\ &= \cos[\alpha] \hat{\mathbf{e}}_x + \cos[\beta] \hat{\mathbf{e}}_y + \cos[\gamma] \hat{\mathbf{e}}_z \\ &= a \hat{\mathbf{e}}_x + b \hat{\mathbf{e}}_y + c \hat{\mathbf{e}}_z \end{aligned}$$

$$\hat{\mathbf{v}} = |\hat{\mathbf{v}}| \hat{\mathbf{u}} = |\hat{\mathbf{v}}| \cos[\alpha] \hat{\mathbf{e}}_x + |\hat{\mathbf{v}}| \cos[\beta] \hat{\mathbf{e}}_y + |\hat{\mathbf{v}}| \cos[\gamma] \hat{\mathbf{e}}_z = v_x \hat{\mathbf{e}}_x + v_y \hat{\mathbf{e}}_y + v_z \hat{\mathbf{e}}_z$$

■ Rotation Matrix: Direction Cosine Matrix



좌표계 {A} 가 z_A 축을 기준으로 α 만큼 회전한 좌표계를 {B} 라고 할 경우 회전행렬 ${}^A_B\mathbf{R}$

$${}^A_B\mathbf{R} = \begin{pmatrix} \hat{x}_A \cdot \hat{x}_B & \hat{x}_A \cdot \hat{y}_B & \hat{x}_A \cdot \hat{z}_B \\ \hat{y}_A \cdot \hat{x}_B & \hat{y}_A \cdot \hat{y}_B & \hat{y}_A \cdot \hat{z}_B \\ \hat{z}_A \cdot \hat{x}_B & \hat{z}_A \cdot \hat{y}_B & \hat{z}_A \cdot \hat{z}_B \end{pmatrix} = \begin{pmatrix} \cos[\alpha] & \cos[90^\circ + \alpha] & \cos[90^\circ] \\ \cos[90^\circ - \alpha] & \cos[\alpha] & \cos[90^\circ] \\ \cos[90^\circ] & \cos[90^\circ] & \cos[0^\circ] \end{pmatrix} = \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{R}_z[\alpha]$$

$$\mathbf{R}_z[\psi] = \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_y[\theta] = \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix}$$

$$\mathbf{R}_x[\phi] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix}$$

■ 회전행렬의 성질 (${}^B\mathbf{R} = {}^A\mathbf{R}^{-1} = {}^A\mathbf{R}^T$)

좌표계 {B} 에 대한 좌표계 {A} 의 회전행렬 : ${}^B_A\mathbf{R}$
 좌표계 {A} 에 대한 좌표계 {B} 의 회전행렬 : ${}^A_B\mathbf{R}$

$$\hat{x}_A = \begin{pmatrix} {}^A x_x \\ {}^A x_y \\ {}^A x_z \end{pmatrix}; \hat{y}_A = \begin{pmatrix} {}^A y_x \\ {}^A y_y \\ {}^A y_z \end{pmatrix}; \hat{z}_A = \begin{pmatrix} {}^A z_x \\ {}^A z_y \\ {}^A z_z \end{pmatrix};$$

$$\hat{x}_B = \begin{pmatrix} {}^B x_x \\ {}^B x_y \\ {}^B x_z \end{pmatrix}; \hat{y}_B = \begin{pmatrix} {}^B y_x \\ {}^B y_y \\ {}^B y_z \end{pmatrix}; \hat{z}_B = \begin{pmatrix} {}^B z_x \\ {}^B z_y \\ {}^B z_z \end{pmatrix};$$

$${}^A_B\mathbf{R} = \begin{pmatrix} \hat{x}_A \cdot \hat{x}_B & \hat{x}_A \cdot \hat{y}_B & \hat{x}_A \cdot \hat{z}_B \\ \hat{y}_A \cdot \hat{x}_B & \hat{y}_A \cdot \hat{y}_B & \hat{y}_A \cdot \hat{z}_B \\ \hat{z}_A \cdot \hat{x}_B & \hat{z}_A \cdot \hat{y}_B & \hat{z}_A \cdot \hat{z}_B \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

$${}^A_B\mathbf{R} = \begin{pmatrix} \hat{x}_A \cdot \hat{x}_B & \hat{x}_A \cdot \hat{y}_B & \hat{x}_A \cdot \hat{z}_B \\ \hat{y}_A \cdot \hat{x}_B & \hat{y}_A \cdot \hat{y}_B & \hat{y}_A \cdot \hat{z}_B \\ \hat{z}_A \cdot \hat{x}_B & \hat{z}_A \cdot \hat{y}_B & \hat{z}_A \cdot \hat{z}_B \end{pmatrix} \Leftrightarrow {}^B_A\mathbf{R} = \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{x}_B \cdot \hat{y}_A & \hat{x}_B \cdot \hat{z}_A \\ \hat{y}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{z}_A \\ \hat{z}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix}$$

$$\begin{cases} {}^A\mathbf{P} = {}^B\mathbf{R} \cdot {}^B\mathbf{P} \\ {}^B\mathbf{P} = {}^A\mathbf{R} \cdot {}^A\mathbf{P} \end{cases} \Rightarrow {}^A\mathbf{P} = {}^B\mathbf{R} \cdot {}^B\mathbf{P} = {}^B\mathbf{R} \cdot {}^A\mathbf{R} \cdot {}^A\mathbf{P} \Rightarrow {}^B\mathbf{R} \cdot {}^A\mathbf{R} = \mathbf{I}$$

$${}^B_A\mathbf{R} = {}^A\mathbf{R}^{-1} = {}^A\mathbf{R}^T$$

■ ${}^A\mathbf{P} = {}^A\mathbf{R} \cdot {}^B\mathbf{P}$

{B} 좌표계 성분을 {A} 좌표계 성분으로 변환

$$\begin{aligned} {}^A\mathbf{P} &= {}^A\mathbf{R} \cdot {}^B\mathbf{P} = \begin{pmatrix} \hat{x}_A \cdot \hat{x}_A & \hat{x}_A \cdot \hat{y}_A & \hat{x}_A \cdot \hat{z}_A \\ \hat{y}_A \cdot \hat{x}_A & \hat{y}_A \cdot \hat{y}_A & \hat{y}_A \cdot \hat{z}_A \\ \hat{z}_A \cdot \hat{x}_A & \hat{z}_A \cdot \hat{y}_A & \hat{z}_A \cdot \hat{z}_A \end{pmatrix} \cdot \begin{pmatrix} {}^A p_x \\ {}^A p_y \\ {}^A p_z \end{pmatrix} = \begin{pmatrix} \hat{x}_A \cdot ({}^A p_x \hat{x}_A + {}^A p_y \hat{y}_A + {}^A p_z \hat{z}_A) \\ \hat{y}_A \cdot ({}^A p_x \hat{x}_A + {}^A p_y \hat{y}_A + {}^A p_z \hat{z}_A) \\ \hat{z}_A \cdot ({}^A p_x \hat{x}_A + {}^A p_y \hat{y}_A + {}^A p_z \hat{z}_A) \end{pmatrix} = \begin{pmatrix} \hat{x}_A \cdot {}^A\mathbf{P} \\ \hat{y}_A \cdot {}^A\mathbf{P} \\ \hat{z}_A \cdot {}^A\mathbf{P} \end{pmatrix} \\ &= \begin{pmatrix} {}^A p_x \\ {}^A p_y \\ {}^A p_z \end{pmatrix} = \begin{pmatrix} \hat{x}_A \cdot {}^B\mathbf{P} \\ \hat{y}_A \cdot {}^B\mathbf{P} \\ \hat{z}_A \cdot {}^B\mathbf{P} \end{pmatrix} = \begin{pmatrix} \hat{x}_A \cdot ({}^B p_x \hat{x}_B + {}^B p_y \hat{y}_B + {}^B p_z \hat{z}_B) \\ \hat{y}_A \cdot ({}^B p_x \hat{x}_B + {}^B p_y \hat{y}_B + {}^B p_z \hat{z}_B) \\ \hat{z}_A \cdot ({}^B p_x \hat{x}_B + {}^B p_y \hat{y}_B + {}^B p_z \hat{z}_B) \end{pmatrix} = \begin{pmatrix} \hat{x}_A \cdot \hat{x}_B & \hat{x}_A \cdot \hat{y}_B & \hat{x}_A \cdot \hat{z}_B \\ \hat{y}_A \cdot \hat{x}_B & \hat{y}_A \cdot \hat{y}_B & \hat{y}_A \cdot \hat{z}_B \\ \hat{z}_A \cdot \hat{x}_B & \hat{z}_A \cdot \hat{y}_B & \hat{z}_A \cdot \hat{z}_B \end{pmatrix} \cdot \begin{pmatrix} {}^B p_x \\ {}^B p_y \\ {}^B p_z \end{pmatrix} = {}^A\mathbf{R} \cdot {}^B\mathbf{P} \end{aligned}$$

→ 좌표계 {B} 에 대해 표현된 점P의 위치 ${}^B\mathbf{P}$ 가 좌표계 {A} 의 좌표값 ${}^A\mathbf{P}$ 로 회전행렬 ${}^A\mathbf{R}$ 을 통하여 변환됨을 의미

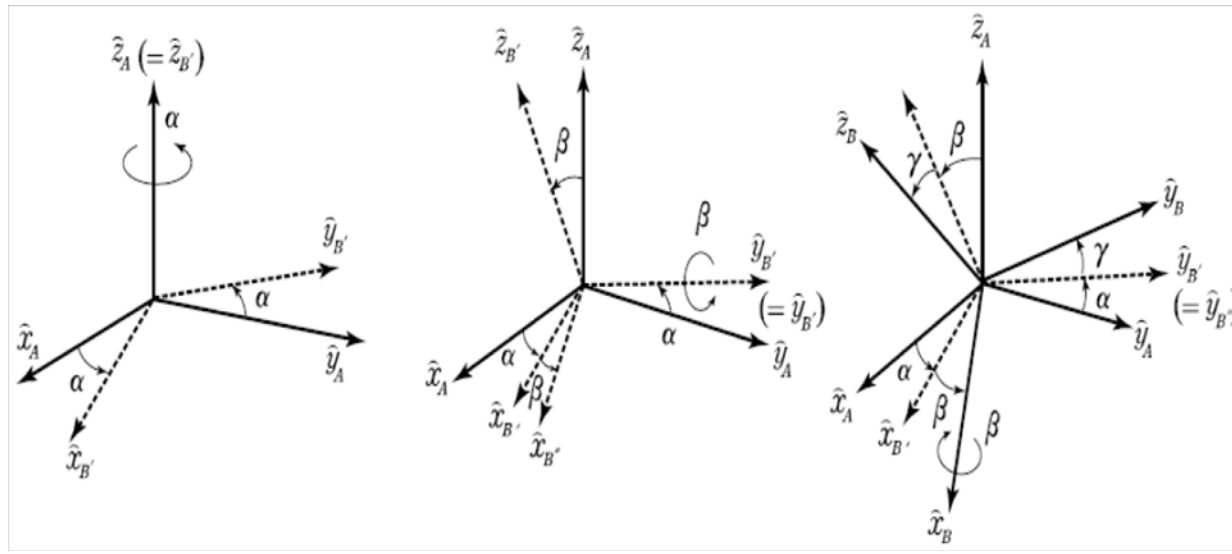
⇒ ${}^A\mathbf{R}$: Transformation Matrix (변환행렬) 라고 정의, 좌표계 {A} 에 대한 좌표계 {B} 의 회전행렬

Euler Angle (Roll ϕ $\hat{x}_B$, Pitch θ $\hat{y}_B$, Yaw ψ $\hat{z}_B$, ZYX) → Rotation Matrix ${}^A_B\mathbf{R}$

■ Z-Y-X 오일러 각 (이동 좌표계)

→ 좌표계 {A} 의 Z축에 대해 ψ 만큼 회전 후
 이동좌표계 즉 회전후 얻어진 새로운 좌표계 {B'} 의 Y축에 대해 θ 만큼 회전한 후,
 다시 새롭게 얻어진 좌표계 (이동좌표계) {B''} 의 X축에 대해 ϕ 만큼 연속적으로 회전을 하여
 특정방위를 나타내는 좌표계 {B} 를 얻는 방법

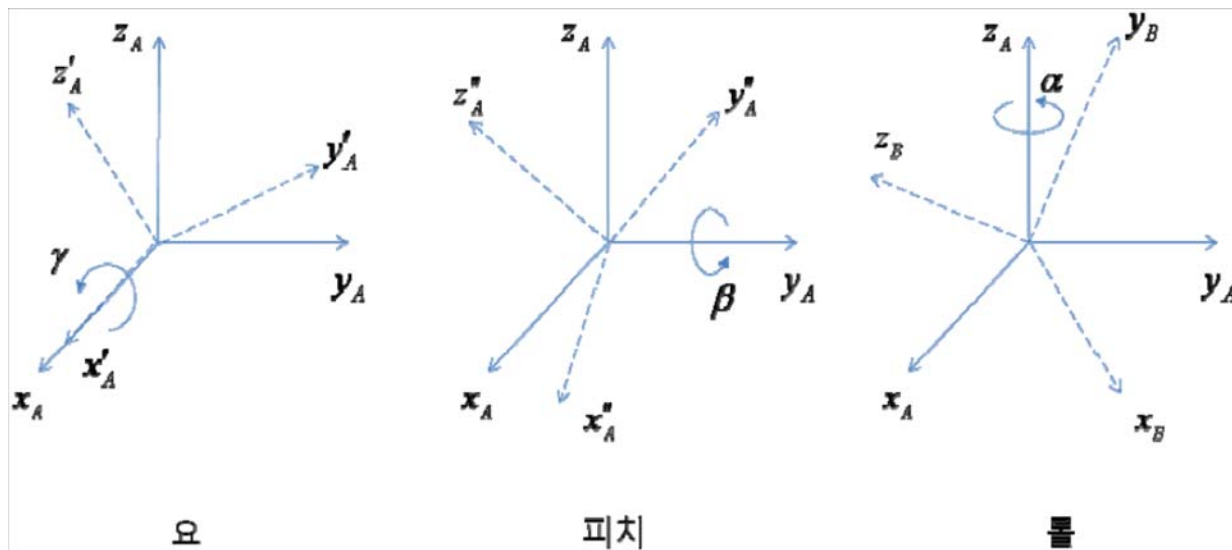
→ 모든 회전이 이동 좌표계에 대한 회전을 이용한 상대변환이므로 해당 변환행렬을
 앞에서부터 순차적으로 곱한다.



■ Roll-Pitch-Yaw (고정 좌표계)

→ 고정기준좌표계 {A} 의 X축에 대해 ϕ 만큼 회전 후
고정좌표계의 Y축에 대해 θ 만큼 회전한 후,
고정좌표계의 Z축에 대해 ψ 만큼 연속적으로 회전을 하여
특정방위를 나타내는 좌표계 {B} 를 얻는 방법

→ 모든 회전이 고정 기준 좌표계에 대한 회전을 이용한 절대변환이므로 해당 변환행렬을
뒤에서부터 앞으로 역순으로 곱한다.



$${}^A_B R = R_z[\psi] R_y[\theta] R_x[\phi]$$

$$= \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix}$$

$$= \begin{pmatrix} \cos[\psi] \cos[\theta] & \cos[\psi] \sin[\theta] \sin[\phi] - \sin[\psi] \cos[\phi] & \cos[\psi] \sin[\theta] \cos[\phi] + \sin[\psi] \sin[\phi] \\ \sin[\psi] \cos[\theta] & \sin[\psi] \sin[\theta] \sin[\phi] + \cos[\psi] \cos[\phi] & \sin[\psi] \sin[\theta] \cos[\phi] - \cos[\psi] \sin[\phi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

■ Rotation Matrix ${}^A_B R \rightarrow$ Roll ϕ $\hat{x}_B$, Pitch θ $\hat{y}_B$, Yaw ψ $\hat{z}_B$

$${}^A_B R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

→ the three RPY angles are $(\theta \in (-\pi/2, \pi/2))$

$$\phi = \text{ArcTan}[r_{33}, r_{32}]$$

$$\theta = \text{ArcTan}[\sqrt{r_{32}^2 + r_{33}^2}, -r_{31}] = \text{ArcSin}[-r_{31}]$$

$$\psi = \text{ArcTan}[r_{11}, r_{21}]$$

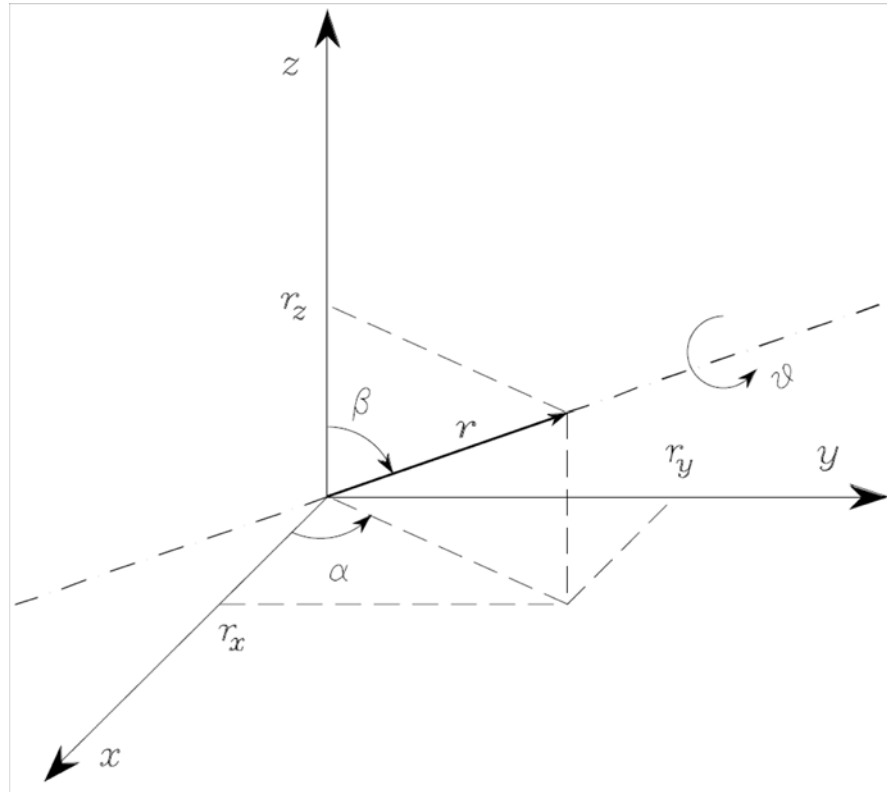
→ or $(\theta \in (\pi/2, 3\pi/2))$

$$\phi = \text{ArcTan}[-r_{33}, -r_{32}]$$

$$\theta = \text{ArcTan}[-\sqrt{r_{32}^2 + r_{33}^2}, -r_{31}] = \text{ArcSin}[-r_{31}]$$

$$\psi = \text{ArcTan}[-r_{11}, -r_{21}]$$

Axis $\mathbf{r}$ /Angle $\vartheta \rightarrow$ Rotation Matrix ${}^A_B\mathbf{R}$



$$\mathbf{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}$$

■ 1

$$\mathbf{R}[\vartheta, \mathbf{r}] = \mathbf{R}_z[\alpha] \cdot \mathbf{R}_y[\beta] \cdot \mathbf{R}_z[\vartheta] \cdot \mathbf{R}_y[-\beta] \cdot \mathbf{R}_z[-\alpha]$$

■ 2

$$\sin[\alpha] = \frac{r_y}{\sqrt{r_x^2 + r_y^2}}; \cos[\alpha] = \frac{r_x}{\sqrt{r_x^2 + r_y^2}};$$

$$\sin[\beta] = \frac{r_z}{\sqrt{r_x^2 + r_y^2 + r_z^2}}; \cos[\beta] = \frac{\sqrt{r_x^2 + r_y^2}}{\sqrt{r_x^2 + r_y^2 + r_z^2}};$$

■ 3

$$\alpha = \text{ArcTan}\left[\frac{r_y}{r_x}\right]; \beta = \text{ArcTan}\left[\frac{\sqrt{r_x^2 + r_y^2}}{r_z}\right];$$

$$\begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[\vartheta] & -\sin[\vartheta] & 0 \\ \sin[\vartheta] & \cos[\vartheta] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[-\beta] & 0 & \sin[-\beta] \\ 0 & 1 & 0 \\ -\sin[-\beta] & 0 & \cos[-\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[-\alpha] & -\sin[-\alpha] & 0 \\ \sin[-\alpha] & \cos[-\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} // \text{Expand} //$$

Factor // Factor // FullSimplify // MatrixForm

■ 4

$$r_x^2 + r_y^2 + r_z^2 = 1;$$

$$\sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z = 1;$$

■ 5

$$\mathbf{R}[\vartheta, \mathbf{r}] = \begin{pmatrix} r_x^2 (1 - \cos[\vartheta]) + \cos[\vartheta] & r_x r_y (1 - \cos[\vartheta]) - r_z \sin[\vartheta] & r_x r_z (1 - \cos[\vartheta]) + r_y \sin[\vartheta] \\ r_x r_y (1 - \cos[\vartheta]) + r_z \sin[\vartheta] & r_y^2 (1 - \cos[\vartheta]) + \cos[\vartheta] & r_y r_z (1 - \cos[\vartheta]) - r_x \sin[\vartheta] \\ r_x r_z (1 - \cos[\vartheta]) - r_y \sin[\vartheta] & r_y r_z (1 - \cos[\vartheta]) + r_x \sin[\vartheta] & r_z^2 (1 - \cos[\vartheta]) + \cos[\vartheta] \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

■ Rotation Matrix ${}^A_B\mathbf{R} \rightarrow$ Axis $\mathbf{r}$ /Angle ϑ

$$\mathbf{R}[-\vartheta, -\mathbf{r}] = \mathbf{R}[\vartheta, \mathbf{r}]$$

$$\vartheta = \text{ArcCos}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right]$$

$$\mathbf{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \frac{1}{2 \sin[\vartheta]} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

$$r_x^2 + r_y^2 + r_z^2 = 1$$

■ Half-Angle Identities

$$\cos\left[\frac{\vartheta}{2}\right] = \sqrt{\frac{1 + \cos[\vartheta]}{2}}$$

$$\sin\left[\frac{\vartheta}{2}\right] = \sqrt{\frac{1 - \cos[\vartheta]}{2}}$$

Orientation ${}^A_B\hat{q} \rightarrow$ Axis r /Angle $\vartheta \rightarrow$ Rotation Matrix ${}^A_BR$

$$\cos\left[\frac{\vartheta}{2}\right] = \pm \sqrt{\frac{1 + \cos[\vartheta]}{2}} ; \quad \sin\left[\frac{\vartheta}{2}\right] = \pm \sqrt{\frac{1 - \cos[\vartheta]}{2}} ;$$

$${}^A_B\hat{q} = (q_1 \quad q_2 \quad q_3 \quad q_4) = \left(\cos\left[\frac{\vartheta}{2}\right] \quad -r_x \sin\left[\frac{\vartheta}{2}\right] \quad -r_y \sin\left[\frac{\vartheta}{2}\right] \quad -r_z \sin\left[\frac{\vartheta}{2}\right] \right)$$

■ 1

$$q_1 = \cos\left[\frac{\vartheta}{2}\right] = \sqrt{\frac{1 + \cos[\vartheta]}{2}}$$

$$q_2 = -r_x \sin\left[\frac{\vartheta}{2}\right] = -r_x \sqrt{\frac{1 - \cos[\vartheta]}{2}}$$

$$q_3 = -r_y \sin\left[\frac{\vartheta}{2}\right] = -r_y \sqrt{\frac{1 - \cos[\vartheta]}{2}}$$

$$q_4 = -r_z \sin\left[\frac{\vartheta}{2}\right] = -r_z \sqrt{\frac{1 - \cos[\vartheta]}{2}}$$

■ 2

$$q_1^2 = \frac{1 + \cos[\vartheta]}{2}$$

$$q_2^2 = r_x^2 \frac{1 - \cos[\vartheta]}{2}$$

$$q_3^2 = r_y^2 \frac{1 - \cos[\vartheta]}{2}$$

$$q_4^2 = r_z^2 \frac{1 - \cos[\vartheta]}{2}$$

■ 3

$$\cos[\vartheta] = 2 q_1^2 - 1$$

$$r_x^2 (1 - \cos[\vartheta]) = 2 q_2^2$$

$$r_y^2 (1 - \cos[\vartheta]) = 2 q_3^2$$

$$r_z^2 (1 - \cos[\vartheta]) = 2 q_4^2$$

■ 4

$$\sin[\vartheta] = \sqrt{1 - \cos[\vartheta]^2} = \sqrt{(1 - \cos[\vartheta]) (1 + \cos[\vartheta])} = \sqrt{2 q_1^2 (1 - \cos[\vartheta])} = \sqrt{2} q_1 \sqrt{1 - \cos[\vartheta]}$$

■ 5

$$r_x = \frac{\sqrt{2} q_2}{\sqrt{1 - \cos[\vartheta]}}$$

$$r_y = \frac{\sqrt{2} q_3}{\sqrt{1 - \cos[\vartheta]}}$$

$$r_z = \frac{\sqrt{2} q_4}{\sqrt{1 - \cos[\vartheta]}}$$

$$\sin[\vartheta] = \sqrt{2} q_1 \sqrt{1 - \cos[\vartheta]}$$

■ 6

$${}^A_B\hat{q} = (q_1 \quad q_2 \quad q_3 \quad q_4) = \left(\cos\left[\frac{\vartheta}{2}\right] \quad -r_x \sin\left[\frac{\vartheta}{2}\right] \quad -r_y \sin\left[\frac{\vartheta}{2}\right] \quad -r_z \sin\left[\frac{\vartheta}{2}\right] \right)$$

$$R[\vartheta, r] = \begin{pmatrix} r_x^2 (1 - \cos[\vartheta]) + \cos[\vartheta] & r_x r_y (1 - \cos[\vartheta]) - r_z \sin[\vartheta] & r_x r_z (1 - \cos[\vartheta]) + r_y \sin[\vartheta] \\ r_x r_y (1 - \cos[\vartheta]) + r_z \sin[\vartheta] & r_y^2 (1 - \cos[\vartheta]) + \cos[\vartheta] & r_y r_z (1 - \cos[\vartheta]) - r_x \sin[\vartheta] \\ r_x r_z (1 - \cos[\vartheta]) - r_y \sin[\vartheta] & r_y r_z (1 - \cos[\vartheta]) + r_x \sin[\vartheta] & r_z^2 (1 - \cos[\vartheta]) + \cos[\vartheta] \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

■ 7

$$\left(\begin{array}{cc} 2 q_2^2 + 2 q_1^2 - 1 & \frac{\sqrt{2} q_2}{\sqrt{1-\cos[\gamma]}} \frac{\sqrt{2} q_3}{\sqrt{1-\cos[\gamma]}} (1 - \cos[\gamma]) + \frac{\sqrt{2} q_4}{\sqrt{1-\cos[\gamma]}} \sqrt{2} q_1 \sqrt{1 - \cos[\gamma]} \\ \frac{\sqrt{2} q_2}{\sqrt{1-\cos[\gamma]}} \frac{\sqrt{2} q_3}{\sqrt{1-\cos[\gamma]}} (1 - \cos[\gamma]) - \frac{\sqrt{2} q_4}{\sqrt{1-\cos[\gamma]}} \sqrt{2} q_1 \sqrt{1 - \cos[\gamma]} & 2 q_3^2 + 2 q_1^2 - 1 \\ \frac{\sqrt{2} q_2}{\sqrt{1-\cos[\gamma]}} \frac{\sqrt{2} q_4}{\sqrt{1-\cos[\gamma]}} (1 - \cos[\gamma]) + \frac{\sqrt{2} q_3}{\sqrt{1-\cos[\gamma]}} \sqrt{2} q_1 \sqrt{1 - \cos[\gamma]} & \frac{\sqrt{2} q_3}{\sqrt{1-\cos[\gamma]}} \frac{\sqrt{2} q_4}{\sqrt{1-\cos[\gamma]}} (1 - \cos[\gamma]) - \frac{\sqrt{2} q_2}{\sqrt{1-\cos[\gamma]}} \sqrt{2} q_1 \sqrt{1 - \cos[\gamma]} \end{array} \right)$$

// FullSimplify // Together // MatrixForm

■ 8

$${}^A_B \mathbf{R} = \left(\begin{array}{ccc} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 + q_1 q_4) & 2 (q_2 q_4 - q_1 q_3) \\ 2 (q_2 q_3 - q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 + q_1 q_2) \\ 2 (q_2 q_4 + q_1 q_3) & 2 (q_3 q_4 - q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{array} \right)$$

■ 9

$$q_1 = \cos\left[\frac{\phi}{2}\right]; q_2 = -r_x \sin\left[\frac{\phi}{2}\right]; q_3 = -r_y \sin\left[\frac{\phi}{2}\right]; q_4 = -r_z \sin\left[\frac{\phi}{2}\right];$$

$$\left(\begin{array}{ccc} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 + q_1 q_4) & 2 (q_2 q_4 - q_1 q_3) \\ 2 (q_2 q_3 - q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 + q_1 q_2) \\ 2 (q_2 q_4 + q_1 q_3) & 2 (q_3 q_4 - q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{array} \right) // \text{FullSimplify} // \text{MatrixForm}$$

$$= \left(\begin{array}{ccc} \cos[\phi] + (1 - \cos[\phi]) r_x^2 & (1 - \cos[\phi]) r_x r_y - \sin[\phi] r_z & (1 - \cos[\phi]) r_x r_z + \sin[\phi] r_y \\ (1 - \cos[\phi]) r_x r_y + \sin[\phi] r_z & \cos[\phi] + (1 - \cos[\phi]) r_y^2 & (1 - \cos[\phi]) r_y r_z - \sin[\phi] r_x \\ (1 - \cos[\phi]) r_x r_z - \sin[\phi] r_y & (1 - \cos[\phi]) r_y r_z + \sin[\phi] r_x & \cos[\phi] + (1 - \cos[\phi]) r_z^2 \end{array} \right)$$

■ Orientation ${}^A_B \hat{\mathbf{q}} \rightarrow \text{Roll } \phi \hat{x}_B, \text{ Pitch } \theta \hat{y}_B, \text{ Yaw } \psi \hat{z}_B$ ϕ around $\hat{x}_B$, θ around $\hat{y}_B$, ψ around $\hat{z}_B$

$${}^A_B \mathbf{R} = \left(\begin{array}{ccc} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{array} \right) = \left(\begin{array}{ccc} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 + q_1 q_4) & 2 (q_2 q_4 - q_1 q_3) \\ 2 (q_2 q_3 - q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 + q_1 q_2) \\ 2 (q_2 q_4 + q_1 q_3) & 2 (q_3 q_4 - q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{array} \right)$$

$$\phi = \text{ArcTan}[{}^A_B \mathbf{R}_{33}, {}^A_B \mathbf{R}_{32}] = \text{ArcTan}[2 q_1^2 - 1 + 2 q_4^2, 2 (q_3 q_4 - q_1 q_2)]$$

$$\theta = -\text{ArcTan}[\sqrt{1 - {}^A_B \mathbf{R}_{31}^2}, {}^A_B \mathbf{R}_{31}] = \text{ArcSin}[-{}^A_B \mathbf{R}_{31}] = -\text{ArcSin}[2 (q_2 q_4 + q_1 q_3)]$$

$$\psi = \text{ArcTan}[{}^A_B \mathbf{R}_{11}, {}^A_B \mathbf{R}_{21}] = \text{ArcTan}[2 q_1^2 - 1 + 2 q_2^2, 2 (q_2 q_3 - q_1 q_4)]$$

Rotation Matrix ${}^A_B \mathbf{R} \rightarrow$ Quaternion ${}^A_B \hat{\mathbf{q}}$

$${}^A_B \mathbf{R} = \left(\begin{array}{ccc} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{array} \right) = \left(\begin{array}{ccc} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 + q_1 q_4) & 2 (q_2 q_4 - q_1 q_3) \\ 2 (q_2 q_3 - q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 + q_1 q_2) \\ 2 (q_2 q_4 + q_1 q_3) & 2 (q_3 q_4 - q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{array} \right)$$

■ 1

$${}^A_B \hat{\mathbf{q}} = (q_1 \quad q_2 \quad q_3 \quad q_4)$$

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

■ 2

$$\begin{aligned} & r_{11} + r_{22} + r_{33} + 1 \\ &= (2 q_1^2 - 1 + 2 q_2^2) + (2 q_1^2 - 1 + 2 q_3^2) + (2 q_1^2 - 1 + 2 q_4^2) + 1 \\ &= 2 (-1 + 2 q_1^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2) = 2 (-1 + 2 q_1^2 + 1) \\ &= 4 q_1^2 \end{aligned}$$

$$\begin{aligned} & r_{11} - r_{22} - r_{33} + 1 \\ &= (2 q_1^2 - 1 + 2 q_2^2) - (2 q_1^2 - 1 + 2 q_3^2) - (2 q_1^2 - 1 + 2 q_4^2) + 1 \\ &= -2 (-1 + q_1^2 - q_2^2 + q_3^2 + q_4^2) = -2 (-1 - 2 q_2^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2) = -2 (-1 - 2 q_2^2 + 1) \\ &= 4 q_2^2 \end{aligned}$$

$$\begin{aligned} & -r_{11} + r_{22} - r_{33} + 1 \\ &= -(2 q_1^2 - 1 + 2 q_2^2) + (2 q_1^2 - 1 + 2 q_3^2) - (2 q_1^2 - 1 + 2 q_4^2) + 1 \\ &= -2 (-1 + q_1^2 + q_2^2 - q_3^2 + q_4^2) = -2 (-1 - 2 q_3^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2) = -2 (-1 - 2 q_3^2 + 1) \\ &= 4 q_3^2 \end{aligned}$$

$$\begin{aligned} & -r_{11} - r_{22} + r_{33} + 1 \\ &= -(2 q_1^2 - 1 + 2 q_2^2) - (2 q_1^2 - 1 + 2 q_3^2) + (2 q_1^2 - 1 + 2 q_4^2) + 1 \\ &= -2 (-1 + q_1^2 + q_2^2 + q_3^2 - q_4^2) = -2 (-1 - 2 q_4^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2) = -2 (-1 - 2 q_4^2 + 1) \\ &= 4 q_4^2 \end{aligned}$$

■ 3

$$\begin{aligned}
q_1 &= \frac{1}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1} \\
q_2 &= -\frac{\text{Sign}[r_{32} - r_{23}]}{2} \sqrt{r_{11} - r_{22} - r_{33} + 1} \\
q_3 &= -\frac{\text{Sign}[r_{13} - r_{31}]}{2} \sqrt{r_{22} - r_{33} - r_{11} + 1} \\
q_4 &= -\frac{\text{Sign}[r_{21} - r_{12}]}{2} \sqrt{r_{33} - r_{11} - r_{22} + 1}
\end{aligned}$$

Euler Angle (Roll $\phi \hat{x}_B$, Pitch $\theta \hat{y}_B$, Yaw $\psi \hat{z}_B$, ZYX) $\rightarrow$ Quaternion ${}^A_B \hat{q}$

$$\begin{aligned}
q_1 &= \frac{1}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1} \\
q_2 &= -\frac{\text{Sign}[r_{32} - r_{23}]}{2} \sqrt{r_{11} - r_{22} - r_{33} + 1} \\
q_3 &= -\frac{\text{Sign}[r_{13} - r_{31}]}{2} \sqrt{r_{22} - r_{33} - r_{11} + 1} \\
q_4 &= -\frac{\text{Sign}[r_{21} - r_{12}]}{2} \sqrt{r_{33} - r_{11} - r_{22} + 1}
\end{aligned}$$

■ 1

$$\begin{aligned}
q_1 &= \frac{1}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1} \\
&= \frac{1}{2} \sqrt{1 + \text{Cos}[\theta] \text{Cos}[\phi] + \text{Cos}[\theta] \text{Cos}[\psi] + \text{Cos}[\phi] \text{Cos}[\psi] + \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi]} \\
q_2 &= -\frac{\text{Sign}[r_{32} - r_{23}]}{2} \sqrt{r_{11} - r_{22} - r_{33} + 1} \\
&= -\frac{1}{2} \text{Sign}[\text{Cos}[\theta] \text{Sin}[\phi] + \text{Cos}[\psi] \text{Sin}[\phi] - \text{Cos}[\phi] \text{Sin}[\theta] \text{Sin}[\psi]] \sqrt{1 - \text{Cos}[\theta] \text{Cos}[\phi] + \text{Cos}[\theta] \text{Cos}[\psi] - \text{Cos}[\phi] \text{Cos}[\psi] - \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi]} \\
q_3 &= -\frac{\text{Sign}[r_{13} - r_{31}]}{2} \sqrt{r_{22} - r_{33} - r_{11} + 1} \\
&= -\frac{1}{2} \text{Sign}[\text{Sin}[\theta] + \text{Cos}[\phi] \text{Cos}[\psi] \text{Sin}[\theta] + \text{Sin}[\phi] \text{Sin}[\psi]] \sqrt{1 - \text{Cos}[\theta] \text{Cos}[\phi] - \text{Cos}[\theta] \text{Cos}[\psi] + \text{Cos}[\phi] \text{Cos}[\psi] + \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi]} \\
q_4 &= -\frac{\text{Sign}[r_{21} - r_{12}]}{2} \sqrt{r_{33} - r_{11} - r_{22} + 1} \\
&= -\frac{1}{2} \text{Sign}[-\text{Cos}[\psi] \text{Sin}[\theta] \text{Sin}[\phi] + \text{Cos}[\theta] \text{Sin}[\psi] + \text{Cos}[\phi] \text{Sin}[\psi]] \sqrt{(1 + \text{Cos}[\theta] \text{Cos}[\phi] - \text{Cos}[\theta] \text{Cos}[\psi] - \text{Cos}[\phi] \text{Cos}[\psi] - \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi])}
\end{aligned}$$

■ 2

$$\begin{aligned}
q_1 &= \frac{1}{2} \sqrt{(1 + \text{Cos}[\theta] \text{Cos}[\phi] + \text{Cos}[\theta] \text{Cos}[\psi] + \text{Cos}[\phi] \text{Cos}[\psi] + \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi])} \\
q_2 &= -\frac{1}{2} \text{Sign}[\text{Cos}[\theta] \text{Sin}[\phi] + \text{Cos}[\psi] \text{Sin}[\phi] - \text{Cos}[\phi] \text{Sin}[\theta] \text{Sin}[\psi]] \sqrt{(1 - \text{Cos}[\theta] \text{Cos}[\phi] + \text{Cos}[\theta] \text{Cos}[\psi] - \text{Cos}[\phi] \text{Cos}[\psi] - \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi])} \\
q_3 &= -\frac{1}{2} \text{Sign}[\text{Sin}[\theta] + \text{Cos}[\phi] \text{Cos}[\psi] \text{Sin}[\theta] + \text{Sin}[\phi] \text{Sin}[\psi]] \sqrt{(1 - \text{Cos}[\theta] \text{Cos}[\phi] - \text{Cos}[\theta] \text{Cos}[\psi] + \text{Cos}[\phi] \text{Cos}[\psi] + \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi])} \\
q_4 &= -\frac{1}{2} \text{Sign}[-\text{Cos}[\psi] \text{Sin}[\theta] \text{Sin}[\phi] + \text{Cos}[\theta] \text{Sin}[\psi] + \text{Cos}[\phi] \text{Sin}[\psi]] \sqrt{(1 + \text{Cos}[\theta] \text{Cos}[\phi] - \text{Cos}[\theta] \text{Cos}[\psi] - \text{Cos}[\phi] \text{Cos}[\psi] - \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi])}
\end{aligned}$$

■ 3

$$\begin{aligned}
Q[\phi_-, \theta_-, \psi_-] &:= \left\{ \frac{1}{2} \sqrt{(1 + \text{Cos}[\theta] \text{Cos}[\phi] + \text{Cos}[\theta] \text{Cos}[\psi] + \text{Cos}[\phi] \text{Cos}[\psi] + \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi])}, \right. \\
&\quad -\frac{1}{2} \text{Sign}[\text{Cos}[\theta] \text{Sin}[\phi] + \text{Cos}[\psi] \text{Sin}[\phi] - \text{Cos}[\phi] \text{Sin}[\theta] \text{Sin}[\psi]] \sqrt{(1 - \text{Cos}[\theta] \text{Cos}[\phi] + \text{Cos}[\theta] \text{Cos}[\psi] - \text{Cos}[\phi] \text{Cos}[\psi] - \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi])}, \\
&\quad -\frac{1}{2} \text{Sign}[\text{Sin}[\theta] + \text{Cos}[\phi] \text{Cos}[\psi] \text{Sin}[\theta] + \text{Sin}[\phi] \text{Sin}[\psi]] \sqrt{(1 - \text{Cos}[\theta] \text{Cos}[\phi] - \text{Cos}[\theta] \text{Cos}[\psi] + \text{Cos}[\phi] \text{Cos}[\psi] + \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi])}, \\
&\quad \left. -\frac{1}{2} \text{Sign}[-\text{Cos}[\psi] \text{Sin}[\theta] \text{Sin}[\phi] + \text{Cos}[\theta] \text{Sin}[\psi] + \text{Cos}[\phi] \text{Sin}[\psi]] \sqrt{(1 + \text{Cos}[\theta] \text{Cos}[\phi] - \text{Cos}[\theta] \text{Cos}[\psi] - \text{Cos}[\phi] \text{Cos}[\psi] - \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi])} \right\}
\end{aligned}$$

$$\begin{aligned}
Q[\phi, \theta, \psi] &= \left\{ \frac{1}{2} \sqrt{2 + 2 \text{Cos}[\phi]}, -\frac{1}{2} \sqrt{2 - 2 \text{Cos}[\phi]} \text{Sign}[\text{Sin}[\phi]], \theta, \psi \right\} \\
&= \left\{ \text{Cos}\left[\frac{\phi}{2}\right], -\text{Sin}\left[\frac{\phi}{2}\right], \theta, \psi \right\}
\end{aligned}$$

$$\begin{aligned}
Q[0, \theta, \psi] &= \left\{ \frac{1}{2} \sqrt{2 + 2 \text{Cos}[\theta]}, 0, -\frac{1}{2} \sqrt{2 - 2 \text{Cos}[\theta]} \text{Sign}[\text{Sin}[\theta]], \psi \right\} \\
&= \left\{ \text{Cos}\left[\frac{\theta}{2}\right], 0, -\text{Sin}\left[\frac{\theta}{2}\right], \psi \right\}
\end{aligned}$$

$$\begin{aligned}
& Q[0, 0, \psi] \\
& = \left\{ \frac{1}{2} \sqrt{2 + 2 \cos[\psi]}, 0, 0, -\frac{1}{2} \sqrt{2 - 2 \cos[\psi]} \operatorname{Sign}[\sin[\psi]] \right\} \\
& = \left\{ \cos\left[\frac{\psi}{2}\right], 0, 0, -\sin\left[\frac{\psi}{2}\right] \right\}
\end{aligned}$$

■ 4

$$\mathbf{B}[\mathbf{a}_-, \mathbf{b}_-] := \text{Flatten}\left[\left(\mathbf{a}[[1]] \quad \mathbf{a}[[2]] \quad \mathbf{a}[[3]] \quad \mathbf{a}[[4]]\right) \cdot \begin{pmatrix} \mathbf{b}[[1]] & \mathbf{b}[[2]] & \mathbf{b}[[3]] & \mathbf{b}[[4]] \\ -\mathbf{b}[[2]] & \mathbf{b}[[1]] & -\mathbf{b}[[4]] & \mathbf{b}[[3]] \\ -\mathbf{b}[[3]] & \mathbf{b}[[4]] & \mathbf{b}[[1]] & -\mathbf{b}[[2]] \\ -\mathbf{b}[[4]] & -\mathbf{b}[[3]] & \mathbf{b}[[2]] & \mathbf{b}[[1]] \end{pmatrix}, 1\right]$$

$$\mathbf{B}[\mathbf{B}[Q[\phi, 0, 0], Q[0, \theta, 0]], Q[0, 0, \psi]]$$

$$\begin{aligned}
& \left\{ \frac{1}{8} \sqrt{2 + 2 \cos[\theta]} \sqrt{2 + 2 \cos[\phi]} \sqrt{2 + 2 \cos[\psi]} + \frac{1}{8} \sqrt{2 - 2 \cos[\theta]} \sqrt{2 - 2 \cos[\phi]} \sqrt{2 - 2 \cos[\psi]} \operatorname{Sign}[\sin[\theta]] \operatorname{Sign}[\sin[\phi]] \operatorname{Sign}[\sin[\psi]], \right. \\
& -\frac{1}{8} \sqrt{2 + 2 \cos[\theta]} \sqrt{2 - 2 \cos[\phi]} \sqrt{2 + 2 \cos[\psi]} \operatorname{Sign}[\sin[\phi]] + \frac{1}{8} \sqrt{2 - 2 \cos[\theta]} \sqrt{2 + 2 \cos[\phi]} \sqrt{2 - 2 \cos[\psi]} \operatorname{Sign}[\sin[\theta]] \operatorname{Sign}[\sin[\psi]], \\
& -\frac{1}{8} \sqrt{2 - 2 \cos[\theta]} \sqrt{2 + 2 \cos[\phi]} \sqrt{2 + 2 \cos[\psi]} \operatorname{Sign}[\sin[\theta]] - \frac{1}{8} \sqrt{2 + 2 \cos[\theta]} \sqrt{2 - 2 \cos[\phi]} \sqrt{2 - 2 \cos[\psi]} \operatorname{Sign}[\sin[\phi]] \operatorname{Sign}[\sin[\psi]], \\
& \left. \frac{1}{8} \sqrt{2 - 2 \cos[\theta]} \sqrt{2 - 2 \cos[\phi]} \sqrt{2 + 2 \cos[\psi]} \operatorname{Sign}[\sin[\theta]] \operatorname{Sign}[\sin[\phi]] - \frac{1}{8} \sqrt{2 + 2 \cos[\theta]} \sqrt{2 + 2 \cos[\phi]} \sqrt{2 - 2 \cos[\psi]} \operatorname{Sign}[\sin[\psi]] \right\}
\end{aligned}$$

■ 5

$$\mathbf{B}\left[\left\{\cos\left[\frac{\phi}{2}\right], -\sin\left[\frac{\phi}{2}\right], 0, 0\right\}, \left\{\cos\left[\frac{\theta}{2}\right], 0, -\sin\left[\frac{\theta}{2}\right], 0\right\}, \left\{\cos\left[\frac{\psi}{2}\right], 0, 0, -\sin\left[\frac{\psi}{2}\right]\right\}\right]$$

$$\begin{aligned}
& \left\{ \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\psi}{2}\right] + \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\psi}{2}\right], -\cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] \sin\left[\frac{\phi}{2}\right] + \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right], \right. \\
& \left. -\cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\psi}{2}\right] \sin\left[\frac{\theta}{2}\right] - \cos\left[\frac{\theta}{2}\right] \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\psi}{2}\right], \cos\left[\frac{\psi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\phi}{2}\right] - \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\psi}{2}\right] \right\}
\end{aligned}$$

$$q_1 = \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\psi}{2}\right] + \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\psi}{2}\right]$$

$$q_2 = -\cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] \sin\left[\frac{\phi}{2}\right] + \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right]$$

$$q_3 = -\cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\psi}{2}\right] \sin\left[\frac{\theta}{2}\right] - \cos\left[\frac{\theta}{2}\right] \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\psi}{2}\right]$$

$$q_4 = \cos\left[\frac{\psi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\phi}{2}\right] - \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\psi}{2}\right]$$

■ 6

$$\{\mathbf{cR} \mathbf{cP} \mathbf{cY} + \mathbf{sR} \mathbf{sP} \mathbf{sY}, -\mathbf{sR} \mathbf{cP} \mathbf{cY} + \mathbf{cR} \mathbf{sP} \mathbf{sY}, -\mathbf{cR} \mathbf{sP} \mathbf{cY} - \mathbf{sR} \mathbf{cP} \mathbf{sY}, \mathbf{sR} \mathbf{sP} \mathbf{cY} - \mathbf{cR} \mathbf{cP} \mathbf{cY}\}$$

$$q_1 = \cos[R/2] \cos[P/2] \cos[Y/2] + \sin[R/2] \sin[P/2] \sin[Y/2]$$

$$q_2 = -\sin[R/2] \cos[P/2] \cos[Y/2] + \cos[R/2] \sin[P/2] \sin[Y/2]$$

$$q_3 = -\cos[R/2] \sin[P/2] \cos[Y/2] - \sin[R/2] \cos[P/2] \sin[Y/2]$$

$$q_4 = \sin[R/2] \sin[P/2] \cos[Y/2] - \cos[R/2] \cos[P/2] \sin[Y/2]$$

Each Angle

$${}^{\mathbf{A}}\mathbf{R} = \mathbf{R}_z[\psi] \mathbf{R}_y[\theta] \mathbf{R}_x[\phi]$$

$$= \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix}$$

$$= \begin{pmatrix} \cos[\psi] \cos[\theta] & \cos[\psi] \sin[\theta] \sin[\phi] - \sin[\psi] \cos[\phi] & \cos[\psi] \sin[\theta] \cos[\phi] + \sin[\psi] \sin[\phi] \\ \sin[\psi] \cos[\theta] & \sin[\psi] \sin[\theta] \sin[\phi] + \cos[\psi] \cos[\phi] & \sin[\psi] \sin[\theta] \cos[\phi] - \cos[\psi] \sin[\phi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

■ 1 YPR

$$\begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$${}^{\mathbf{A}}\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 + q_1 q_4) & 2 (q_2 q_4 - q_1 q_3) \\ 2 (q_2 q_3 - q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 + q_1 q_2) \\ 2 (q_2 q_4 + q_1 q_3) & 2 (q_3 q_4 - q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{pmatrix}$$

$$\phi = \operatorname{ArcTan}[{}^{\mathbf{A}}\mathbf{R}_{33}, {}^{\mathbf{A}}\mathbf{R}_{32}] = \operatorname{ArcTan}[2 q_1^2 - 1 + 2 q_4^2, 2 (q_3 q_4 - q_1 q_2)]$$

$$\theta = \operatorname{ArcSin}[-{}^{\mathbf{A}}\mathbf{R}_{31}] = -\operatorname{ArcSin}[2 (q_2 q_4 + q_1 q_3)]$$

$$\psi = \operatorname{ArcTan}[{}^{\mathbf{A}}\mathbf{R}_{11}, {}^{\mathbf{A}}\mathbf{R}_{21}] = \operatorname{ArcTan}[2 q_1^2 - 1 + 2 q_2^2, 2 (q_2 q_3 - q_1 q_4)]$$

- 2 RYP

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} \cdot \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] \cos[\psi] & -\sin[\psi] & \cos[\psi] \sin[\theta] \\ \sin[\theta] \sin[\phi] + \cos[\theta] \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] & -\cos[\theta] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\cos[\phi] \sin[\theta] + \cos[\theta] \sin[\phi] \sin[\psi] & \cos[\psi] \sin[\phi] & \cos[\theta] \cos[\phi] + \sin[\theta] \sin[\phi] \sin[\psi] \end{pmatrix}$$

$${}^{\text{A}}_{\text{B}}\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} 2 \, q_2^2 - 1 + 2 \, q_2^2 & 2 \, (q_2 \, q_3 + q_1 \, q_4) & 2 \, (q_2 \, q_4 - q_1 \, q_3) \\ 2 \, (q_2 \, q_3 - q_1 \, q_4) & 2 \, q_1^2 - 1 + 2 \, q_3^2 & 2 \, (q_3 \, q_4 + q_1 \, q_2) \\ 2 \, (q_2 \, q_4 + q_1 \, q_3) & 2 \, (q_3 \, q_4 - q_1 \, q_2) & 2 \, q_1^2 - 1 + 2 \, q_4^2 \end{pmatrix}$$

$$\phi = \text{ArcTan}[{}^{\text{A}}_{\text{B}}\mathbf{R}_{22}, {}^{\text{A}}_{\text{B}}\mathbf{R}_{32}] = \text{ArcTan}[2 \, q_1^2 - 1 + 2 \, q_3^2, 2 \, (q_3 \, q_4 - q_1 \, q_2)]$$

$$\theta = \text{ArcTan}[{}^{\text{A}}_{\text{B}}\mathbf{R}_{11}, {}^{\text{A}}_{\text{B}}\mathbf{R}_{13}] = \text{ArcTan}[2 \, q_1^2 - 1 + 2 \, q_2^2, 2 \, (q_2 \, q_4 - q_1 \, q_3)]$$

$$\psi = \text{ArcSin}[-{}^{\text{A}}_{\text{B}}\mathbf{R}_{12}] = \text{ArcSin}[-2 \, (q_2 \, q_3 + q_1 \, q_4)]$$

■ 3 PRY

$$\begin{pmatrix} \text{Cos}[\theta] & 0 & \text{Sin}[\theta] \\ 0 & 1 & 0 \\ -\text{Sin}[\theta] & 0 & \text{Cos}[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[\phi] & -\text{Sin}[\phi] \\ 0 & \text{Sin}[\phi] & \text{Cos}[\phi] \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}[\psi] & -\text{Sin}[\psi] & 0 \\ \text{Sin}[\psi] & \text{Cos}[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \text{Cos}[\theta] \text{Cos}[\psi] + \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi] & \text{Cos}[\psi] \text{Sin}[\theta] \text{Sin}[\phi] - \text{Cos}[\theta] \text{Sin}[\psi] & \text{Cos}[\phi] \text{Sin}[\theta] \\ \text{Cos}[\phi] \text{Sin}[\psi] & \text{Cos}[\phi] \text{Cos}[\psi] & -\text{Sin}[\phi] \\ -\text{Cos}[\psi] \text{Sin}[\theta] + \text{Cos}[\theta] \text{Sin}[\phi] \text{Sin}[\psi] & \text{Cos}[\theta] \text{Cos}[\psi] \text{Sin}[\phi] + \text{Sin}[\theta] \text{Sin}[\psi] & \text{Cos}[\theta] \text{Cos}[\phi] \end{pmatrix}$$

$${}^A_B \mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 + q_1 q_4) & 2 (q_2 q_4 - q_1 q_3) \\ 2 (q_2 q_3 - q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 + q_1 q_2) \\ 2 (q_2 q_4 + q_1 q_3) & 2 (q_3 q_4 - q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{pmatrix}$$

$$\phi = \text{ArcSin}[-{}^A_B R_{23}] = \text{ArcSin}[-2 (q_3 q_4 + q_1 q_2)]$$

$$\theta = \text{ArcTan}[{}^A_B R_{33}, {}^A_B R_{13}] = \text{ArcTan}[2 q_1^2 - 1 + 2 q_4^2, 2 (q_2 q_4 - q_1 q_3)]$$

$$\psi = \text{ArcTan}[{}^A_B R_{22}, {}^A_B R_{21}] = \text{ArcTan}[2 q_1^2 - 1 + 2 q_3^2, 2 (q_2 q_3 - q_1 q_4)]$$

■ 4 PYR

$$\begin{pmatrix} \text{Cos}[\theta] & 0 & \text{Sin}[\theta] \\ 0 & 1 & 0 \\ -\text{Sin}[\theta] & 0 & \text{Cos}[\theta] \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}[\psi] & -\text{Sin}[\psi] & 0 \\ \text{Sin}[\psi] & \text{Cos}[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[\phi] & -\text{Sin}[\phi] \\ 0 & \text{Sin}[\phi] & \text{Cos}[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \text{Cos}[\theta] \text{Cos}[\psi] & \text{Sin}[\theta] \text{Sin}[\phi] - \text{Cos}[\theta] \text{Cos}[\phi] \text{Sin}[\psi] & \text{Cos}[\phi] \text{Sin}[\theta] + \text{Cos}[\theta] \text{Sin}[\phi] \text{Sin}[\psi] \\ \text{Sin}[\psi] & \text{Cos}[\phi] \text{Cos}[\psi] & -\text{Cos}[\psi] \text{Sin}[\phi] \\ -\text{Cos}[\psi] \text{Sin}[\theta] & \text{Cos}[\theta] \text{Sin}[\phi] + \text{Cos}[\phi] \text{Sin}[\theta] \text{Sin}[\psi] & \text{Cos}[\theta] \text{Cos}[\phi] - \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi] \end{pmatrix}$$

$${}^A_B \mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} 2 \, q_1^2 - 1 + 2 \, q_2^2 & 2 \, (q_2 \, q_3 + q_1 \, q_4) & 2 \, (q_2 \, q_4 - q_1 \, q_3) \\ 2 \, (q_2 \, q_3 - q_1 \, q_4) & 2 \, q_1^2 - 1 + 2 \, q_3^2 & 2 \, (q_3 \, q_4 + q_1 \, q_2) \\ 2 \, (q_2 \, q_4 + q_1 \, q_3) & 2 \, (q_3 \, q_4 - q_1 \, q_2) & 2 \, q_1^2 - 1 + 2 \, q_4^2 \end{pmatrix}$$

$$\phi = \text{ArcTan}[\hat{R}_{22}, -\hat{R}_{23}] = \text{ArcTan}[2 \, q_1^2 - 1 + 2 \, q_3^2, -2 \, (q_3 \, q_4 + q_1 \, q_2)]$$

$$\theta = \text{ArcTan}[\hat{R}_{11}, -\hat{R}_{31}] = \text{ArcTan}[2 \, q_1^2 - 1 + 2 \, q_2^2, -2 \, (q_2 \, q_4 + q_1 \, q_3)]$$

$$\psi = \text{ArcSin}[\hat{R}_{21}] = \text{ArcSin}[2 \, (q_2 \, q_3 - q_1 \, q_4)]$$

■ 5 YRP

$$\begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\phi] \sin[\psi] & \cos[\psi] \sin[\theta] + \cos[\theta] \sin[\phi] \sin[\psi] \\ \cos[\psi] \sin[\theta] \sin[\phi] + \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] & -\cos[\theta] \cos[\psi] \sin[\phi] + \sin[\theta] \sin[\psi] \\ -\cos[\phi] \sin[\theta] & \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$${}^A_B\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} 2 \, q_1^2 - 1 + 2 \, q_2^2 & 2 \, (q_2 \, q_3 + q_1 \, q_4) & 2 \, (q_2 \, q_4 - q_1 \, q_3) \\ 2 \, (q_2 \, q_3 - q_1 \, q_4) & 2 \, q_1^2 - 1 + 2 \, q_3^2 & 2 \, (q_3 \, q_4 + q_1 \, q_2) \\ 2 \, (q_2 \, q_4 + q_1 \, q_3) & 2 \, (q_3 \, q_4 - q_1 \, q_2) & 2 \, q_1^2 - 1 + 2 \, q_4^2 \end{pmatrix}$$

$$\phi = \text{ArcSin}[{}^A_B\mathbf{R}_{32}] = \text{ArcSin}[2 \, (q_3 \, q_4 - q_1 \, q_2)]$$

$$\theta = \text{ArcTan}[{}^A_B\mathbf{R}_{33}, -{}^A_B\mathbf{R}_{31}] = \text{ArcTan}[2 \, q_1^2 - 1 + 2 \, q_4^2, -2 \, (q_2 \, q_4 + q_1 \, q_3)]$$

$$\psi = \text{ArcTan}[{}^A_B\mathbf{R}_{22}, -{}^A_B\mathbf{R}_{12}] = \text{ArcTan}[2 \, q_1^2 - 1 + 2 \, q_3^2, -2 \, (q_2 \, q_3 + q_1 \, q_4)]$$

- **6 RPY**

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] \cos[\psi] & -\cos[\theta] \sin[\psi] & \sin[\theta] \\ \cos[\psi] \sin[\theta] \sin[\phi] + \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\theta] \sin[\phi] \\ -\cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] & \cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$${}^A_B\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} 2q_1^2 - 1 + 2q_2^2 & 2(q_2q_3 + q_1q_4) & 2(q_2q_4 - q_1q_3) \\ 2(q_2q_3 - q_1q_4) & 2q_1^2 - 1 + 2q_3^2 & 2(q_3q_4 + q_1q_2) \\ 2(q_2q_4 + q_1q_3) & 2(q_3q_4 - q_1q_2) & 2q_1^2 - 1 + 2q_4^2 \end{pmatrix}$$

$$\phi = \text{ArcTan}[{}^A_B\mathbf{R}_{33}, -{}^A_B\mathbf{R}_{23}] = \text{ArcTan}[2q_1^2 - 1 + 2q_4^2, -2(q_3q_4 + q_1q_2)]$$

$$\theta = \text{ArcSin}[{}^A_B\mathbf{R}_{13}] = \text{ArcSin}[2(q_2q_4 - q_1q_3)]$$

$$\psi = \text{ArcTan}[{}^A_B\mathbf{R}_{11}, -{}^A_B\mathbf{R}_{12}] = \text{ArcTan}[2q_1^2 - 1 + 2q_2^2, -2(q_2q_3 + q_1q_4)]$$

Filter derivation

Orientation from angular rate

- A tri-axis gyroscope

- The angular rate

$$^S\omega = (\theta \quad \omega_x \quad \omega_y \quad \omega_z)$$

- The quaternion derivative

$$\dot{\hat{\mathbf{q}}} = \frac{1}{2} \hat{\mathbf{q}} \otimes ^S\omega$$

- The quaternion derivative measured at time t

$$\dot{\hat{\mathbf{q}}}_{\omega, t} = \frac{1}{2} \hat{\mathbf{q}}_{\text{est}, t-1} \otimes ^S\omega_t$$

$$(\hat{q}_{\omega 1} \quad \hat{q}_{\omega 2} \quad \hat{q}_{\omega 3} \quad \hat{q}_{\omega 4}) = \frac{1}{2} (q_1 \quad q_2 \quad q_3 \quad q_4) \otimes (\theta \quad \omega_x \quad \omega_y \quad \omega_z)$$

- The orientation of the earth frame at time t

$$\hat{\mathbf{q}}_{\omega, t} = \hat{\mathbf{q}}_{\text{est}, t-1} + \dot{\hat{\mathbf{q}}}_{\omega, t} \Delta t$$

The samling period Δt

$$(q_1 \quad q_2 \quad q_3 \quad q_4) = (\hat{q}_1 \quad \hat{q}_2 \quad \hat{q}_3 \quad \hat{q}_4) + \left((\hat{q}_{\omega 1} \quad \hat{q}_{\omega 2} \quad \hat{q}_{\omega 3} \quad \hat{q}_{\omega 4}) - \beta \left(\dot{\hat{q}}_1 \quad \dot{\hat{q}}_2 \quad \dot{\hat{q}}_3 \quad \dot{\hat{q}}_4 \right) \right) * \Delta t$$

Orientation from vector observations

- A tri-axis accelerometer (Linear accelerations due to motion)

the field of gravity

- A tri-axis magnetometer (Local magnetic flux and distortions)

the earth ' s magnetic field

- A quaternion

$$\hat{\mathbf{q}} = (q_1 \quad q_2 \quad q_3 \quad q_4)$$

- A predefined reference direction

$$^E\mathbf{d} = (\theta \quad d_x \quad d_y \quad d_z)$$

- The measured direction

$$^S\mathbf{s} = (\theta \quad s_x \quad s_y \quad s_z)$$

- The objective function

$$f[\hat{\mathbf{q}}, ^E\mathbf{d}, ^S\mathbf{s}] = \hat{\mathbf{q}}^* \otimes ^E\mathbf{d} \otimes \hat{\mathbf{q}} - ^S\mathbf{s}$$

- A quaternion may be found

$$\min_{\hat{\mathbf{q}} \in \mathbb{R}^4} f[\hat{\mathbf{q}}, ^E\mathbf{d}, ^S\mathbf{s}]$$

The gradient descent algorithm

■ Linear system

$$\mathbf{Ax} - \mathbf{b} = \mathbf{0}$$

$$\mathbf{F}[\mathbf{x}] = \|\mathbf{Ax} - \mathbf{b}\|^2$$

$$\nabla \mathbf{F}[\mathbf{x}] = 2 \mathbf{A}^T (\mathbf{Ax} - \mathbf{b})$$

$$\mathbf{x} = \mathbf{x} - \gamma \nabla \mathbf{F}[\mathbf{x}]$$

$$\nabla \mathbf{F}[\mathbf{x}] = \mathbf{J}^T \mathbf{F}[\mathbf{x}]$$

■ An orientation estimation of $\hat{\mathbf{q}}_{n+1}$

$$\hat{\mathbf{q}}_{k+1} = \hat{\mathbf{q}}_k - \mu \frac{\nabla f[\hat{\mathbf{q}}, \mathbf{d}, \mathbf{s}]}{\|\nabla f[\hat{\mathbf{q}}, \mathbf{d}, \mathbf{s}]\|}, \quad k = 0, 1, \dots, n$$

A step-size μ

■ The gradient of the solution surface (General form)

$$\nabla f[\hat{\mathbf{q}}_k, \mathbf{d}, \mathbf{s}] = \mathbf{J}^T[\hat{\mathbf{q}}_k, \mathbf{d}] f[\hat{\mathbf{q}}_k, \mathbf{d}, \mathbf{s}]$$

■ A objective function

$$f[\hat{\mathbf{q}}, \mathbf{d}, \mathbf{s}] = \hat{\mathbf{q}}^* \otimes \mathbf{d} \otimes \hat{\mathbf{q}} - \mathbf{s}$$

$$\hat{\mathbf{q}} = (\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \quad \mathbf{q}_4)$$

$$\mathbf{d} = (\mathbf{0} \quad \mathbf{d}_x \quad \mathbf{d}_y \quad \mathbf{d}_z)$$

$$\mathbf{s} = (\mathbf{0} \quad \mathbf{s}_x \quad \mathbf{s}_y \quad \mathbf{s}_z)$$

$$\hat{\mathbf{q}}^* \otimes \mathbf{d} \otimes \hat{\mathbf{q}} = (\mathbf{q}_1 \quad -\mathbf{q}_2 \quad -\mathbf{q}_3 \quad -\mathbf{q}_4) \otimes (\mathbf{0} \quad \mathbf{d}_x \quad \mathbf{d}_y \quad \mathbf{d}_z) \otimes (\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \quad \mathbf{q}_4)$$

$$= (\mathbf{q}_1 \quad -\mathbf{q}_2 \quad -\mathbf{q}_3 \quad -\mathbf{q}_4) \cdot \begin{pmatrix} \mathbf{0} & \mathbf{d}_x & \mathbf{d}_y & \mathbf{d}_z \\ -\mathbf{d}_x & \mathbf{0} & -\mathbf{d}_z & \mathbf{d}_y \\ -\mathbf{d}_y & \mathbf{d}_z & \mathbf{0} & -\mathbf{d}_x \\ -\mathbf{d}_z & -\mathbf{d}_y & \mathbf{d}_x & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 \\ -\mathbf{q}_2 & \mathbf{q}_1 & -\mathbf{q}_4 & \mathbf{q}_3 \\ -\mathbf{q}_3 & \mathbf{q}_4 & \mathbf{q}_1 & -\mathbf{q}_2 \\ -\mathbf{q}_4 & -\mathbf{q}_3 & \mathbf{q}_2 & \mathbf{q}_1 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{0} \\ 2 \mathbf{d}_y (\mathbf{q}_2 \mathbf{q}_3 + \mathbf{q}_1 \mathbf{q}_4) + \mathbf{d}_z (-2 \mathbf{q}_1 \mathbf{q}_3 + 2 \mathbf{q}_2 \mathbf{q}_4) + \mathbf{d}_x (\mathbf{q}_1^2 + \mathbf{q}_2^2 - \mathbf{q}_3^2 - \mathbf{q}_4^2) \\ 2 \mathbf{d}_x (\mathbf{q}_2 \mathbf{q}_3 - \mathbf{q}_1 \mathbf{q}_4) + 2 \mathbf{d}_z (\mathbf{q}_1 \mathbf{q}_2 + \mathbf{q}_3 \mathbf{q}_4) + \mathbf{d}_y (\mathbf{q}_1^2 - \mathbf{q}_2^2 + \mathbf{q}_3^2 - \mathbf{q}_4^2) \\ 2 \mathbf{d}_x (\mathbf{q}_1 \mathbf{q}_3 + \mathbf{q}_2 \mathbf{q}_4) + \mathbf{d}_y (-2 \mathbf{q}_1 \mathbf{q}_2 + 2 \mathbf{q}_3 \mathbf{q}_4) + \mathbf{d}_z (\mathbf{q}_1^2 - \mathbf{q}_2^2 - \mathbf{q}_3^2 + \mathbf{q}_4^2) \end{pmatrix}^T$$

$$= \begin{pmatrix} \mathbf{0} \\ 2 (\mathbf{d}_x (\frac{1}{2} - \mathbf{q}_3^2 - \mathbf{q}_4^2) + \mathbf{d}_y (\mathbf{q}_2 \mathbf{q}_3 + \mathbf{q}_1 \mathbf{q}_4) + \mathbf{d}_z (\mathbf{q}_2 \mathbf{q}_4 - \mathbf{q}_1 \mathbf{q}_3)) \\ 2 (\mathbf{d}_x (\mathbf{q}_2 \mathbf{q}_3 - \mathbf{q}_1 \mathbf{q}_4) + \mathbf{d}_y (\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_4^2) + \mathbf{d}_z (\mathbf{q}_3 \mathbf{q}_4 + \mathbf{q}_1 \mathbf{q}_2)) \\ 2 (\mathbf{d}_x (\mathbf{q}_2 \mathbf{q}_4 + \mathbf{q}_1 \mathbf{q}_3) + \mathbf{d}_y (\mathbf{q}_3 \mathbf{q}_4 - \mathbf{q}_1 \mathbf{q}_2) + \mathbf{d}_z (\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_3^2)) \end{pmatrix}^T$$

$$f[\hat{\mathbf{q}}_k, \mathbf{d}, \mathbf{s}] = \begin{pmatrix} 2 \mathbf{d}_x (\frac{1}{2} - \mathbf{q}_3^2 - \mathbf{q}_4^2) + 2 \mathbf{d}_y (\mathbf{q}_2 \mathbf{q}_3 + \mathbf{q}_1 \mathbf{q}_4) + 2 \mathbf{d}_z (\mathbf{q}_2 \mathbf{q}_4 - \mathbf{q}_1 \mathbf{q}_3) - \mathbf{s}_x \\ 2 \mathbf{d}_x (\mathbf{q}_2 \mathbf{q}_3 - \mathbf{q}_1 \mathbf{q}_4) + 2 \mathbf{d}_y (\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_4^2) + 2 \mathbf{d}_z (\mathbf{q}_3 \mathbf{q}_4 + \mathbf{q}_1 \mathbf{q}_2) - \mathbf{s}_y \\ 2 \mathbf{d}_x (\mathbf{q}_2 \mathbf{q}_4 + \mathbf{q}_1 \mathbf{q}_3) + 2 \mathbf{d}_y (\mathbf{q}_3 \mathbf{q}_4 - \mathbf{q}_1 \mathbf{q}_2) + 2 \mathbf{d}_z (\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_3^2) - \mathbf{s}_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}[\hat{\mathbf{q}}_k, \mathbf{d}]$$

$$= \begin{pmatrix} 2 \mathbf{d}_y \mathbf{q}_4 - 2 \mathbf{d}_z \mathbf{q}_3 & 2 \mathbf{d}_y \mathbf{q}_3 + 2 \mathbf{d}_z \mathbf{q}_4 & -4 \mathbf{d}_x \mathbf{q}_3 + 2 \mathbf{d}_y \mathbf{q}_2 - 2 \mathbf{d}_z \mathbf{q}_1 & -4 \mathbf{d}_x \mathbf{q}_4 + 2 \mathbf{d}_y \mathbf{q}_1 + 2 \mathbf{d}_z \mathbf{q}_2 \\ -2 \mathbf{d}_x \mathbf{q}_4 + 2 \mathbf{d}_z \mathbf{q}_2 & 2 \mathbf{d}_x \mathbf{q}_3 - 4 \mathbf{d}_y \mathbf{q}_2 + 2 \mathbf{d}_z \mathbf{q}_1 & 2 \mathbf{d}_x \mathbf{q}_2 + 2 \mathbf{d}_z \mathbf{q}_4 & -2 \mathbf{d}_x \mathbf{q}_1 - 4 \mathbf{d}_y \mathbf{q}_4 + 2 \mathbf{d}_z \mathbf{q}_3 \\ 2 \mathbf{d}_x \mathbf{q}_3 - 2 \mathbf{d}_y \mathbf{q}_2 & 2 \mathbf{d}_x \mathbf{q}_4 - 2 \mathbf{d}_y \mathbf{q}_1 - 4 \mathbf{d}_z \mathbf{q}_2 & 2 \mathbf{d}_x \mathbf{q}_1 + 2 \mathbf{d}_y \mathbf{q}_4 - 4 \mathbf{d}_z \mathbf{q}_3 & 2 \mathbf{d}_x \mathbf{q}_2 + 2 \mathbf{d}_y \mathbf{q}_3 \end{pmatrix}$$

$$\mathbf{J}[\hat{\mathbf{q}}_k, \mathbf{d}] = \begin{pmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \frac{\partial f_1}{\partial q_3} & \frac{\partial f_1}{\partial q_4} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \frac{\partial f_2}{\partial q_3} & \frac{\partial f_2}{\partial q_4} \\ \frac{\partial f_3}{\partial q_1} & \frac{\partial f_3}{\partial q_2} & \frac{\partial f_3}{\partial q_3} & \frac{\partial f_3}{\partial q_4} \end{pmatrix} = \begin{pmatrix} \partial_{q_1} f_1 & \partial_{q_2} f_1 & \partial_{q_3} f_1 & \partial_{q_4} f_1 \\ \partial_{q_1} f_2 & \partial_{q_2} f_2 & \partial_{q_3} f_2 & \partial_{q_4} f_2 \\ \partial_{q_1} f_3 & \partial_{q_2} f_3 & \partial_{q_3} f_3 & \partial_{q_4} f_3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \mathbf{d}_y \mathbf{q}_4 - 2 \mathbf{d}_z \mathbf{q}_3 & 2 \mathbf{d}_y \mathbf{q}_3 + 2 \mathbf{d}_z \mathbf{q}_4 & -4 \mathbf{d}_x \mathbf{q}_3 + 2 \mathbf{d}_y \mathbf{q}_2 - 2 \mathbf{d}_z \mathbf{q}_1 & -4 \mathbf{d}_x \mathbf{q}_4 + 2 \mathbf{d}_y \mathbf{q}_1 + 2 \mathbf{d}_z \mathbf{q}_2 \\ -2 \mathbf{d}_x \mathbf{q}_4 + 2 \mathbf{d}_z \mathbf{q}_2 & 2 \mathbf{d}_x \mathbf{q}_3 - 4 \mathbf{d}_y \mathbf{q}_2 + 2 \mathbf{d}_z \mathbf{q}_1 & 2 \mathbf{d}_x \mathbf{q}_2 + 2 \mathbf{d}_z \mathbf{q}_4 & -2 \mathbf{d}_x \mathbf{q}_1 - 4 \mathbf{d}_y \mathbf{q}_4 + 2 \mathbf{d}_z \mathbf{q}_3 \\ 2 \mathbf{d}_x \mathbf{q}_3 - 2 \mathbf{d}_y \mathbf{q}_2 & 2 \mathbf{d}_x \mathbf{q}_4 - 2 \mathbf{d}_y \mathbf{q}_1 - 4 \mathbf{d}_z \mathbf{q}_2 & 2 \mathbf{d}_x \mathbf{q}_1 + 2 \mathbf{d}_y \mathbf{q}_4 - 4 \mathbf{d}_z \mathbf{q}_3 & 2 \mathbf{d}_x \mathbf{q}_2 + 2 \mathbf{d}_y \mathbf{q}_3 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} = \begin{pmatrix} 2 \mathbf{d}_x \left(\frac{1}{2} - \mathbf{q}_3^2 - \mathbf{q}_4^2 \right) + 2 \mathbf{d}_y \left(\mathbf{q}_2 \mathbf{q}_3 + \mathbf{q}_1 \mathbf{q}_4 \right) + 2 \mathbf{d}_z \left(\mathbf{q}_2 \mathbf{q}_4 - \mathbf{q}_1 \mathbf{q}_3 \right) - \mathbf{s}_x \\ 2 \mathbf{d}_x \left(\mathbf{q}_2 \mathbf{q}_3 - \mathbf{q}_1 \mathbf{q}_4 \right) + 2 \mathbf{d}_y \left(\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_4^2 \right) + 2 \mathbf{d}_z \left(\mathbf{q}_3 \mathbf{q}_4 + \mathbf{q}_1 \mathbf{q}_2 \right) - \mathbf{s}_y \\ 2 \mathbf{d}_x \left(\mathbf{q}_2 \mathbf{q}_4 + \mathbf{q}_1 \mathbf{q}_3 \right) + 2 \mathbf{d}_y \left(\mathbf{q}_3 \mathbf{q}_4 - \mathbf{q}_1 \mathbf{q}_2 \right) + 2 \mathbf{d}_z \left(\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_3^2 \right) - \mathbf{s}_z \end{pmatrix};$$

$$\begin{pmatrix} \partial_{\mathbf{q}_1} \mathbf{f}_1 & \partial_{\mathbf{q}_2} \mathbf{f}_1 & \partial_{\mathbf{q}_3} \mathbf{f}_1 & \partial_{\mathbf{q}_4} \mathbf{f}_1 \\ \partial_{\mathbf{q}_1} \mathbf{f}_2 & \partial_{\mathbf{q}_2} \mathbf{f}_2 & \partial_{\mathbf{q}_3} \mathbf{f}_2 & \partial_{\mathbf{q}_4} \mathbf{f}_2 \\ \partial_{\mathbf{q}_1} \mathbf{f}_3 & \partial_{\mathbf{q}_2} \mathbf{f}_3 & \partial_{\mathbf{q}_3} \mathbf{f}_3 & \partial_{\mathbf{q}_4} \mathbf{f}_3 \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} -2 \mathbf{d}_z \mathbf{q}_3 + 2 \mathbf{d}_y \mathbf{q}_4 & 2 \mathbf{d}_y \mathbf{q}_3 + 2 \mathbf{d}_z \mathbf{q}_4 & -2 \mathbf{d}_z \mathbf{q}_1 + 2 \mathbf{d}_y \mathbf{q}_2 - 4 \mathbf{d}_x \mathbf{q}_3 & 2 \mathbf{d}_y \mathbf{q}_1 + 2 \mathbf{d}_z \mathbf{q}_2 - 4 \mathbf{d}_x \mathbf{q}_4 \\ 2 \mathbf{d}_z \mathbf{q}_2 - 2 \mathbf{d}_x \mathbf{q}_4 & 2 \mathbf{d}_z \mathbf{q}_1 - 4 \mathbf{d}_y \mathbf{q}_2 + 2 \mathbf{d}_x \mathbf{q}_3 & 2 \mathbf{d}_x \mathbf{q}_2 + 2 \mathbf{d}_z \mathbf{q}_4 & -2 \mathbf{d}_x \mathbf{q}_1 + 2 \mathbf{d}_z \mathbf{q}_3 - 4 \mathbf{d}_y \mathbf{q}_4 \\ -2 \mathbf{d}_y \mathbf{q}_2 + 2 \mathbf{d}_x \mathbf{q}_3 & -2 \mathbf{d}_y \mathbf{q}_1 - 4 \mathbf{d}_z \mathbf{q}_2 + 2 \mathbf{d}_x \mathbf{q}_4 & 2 \mathbf{d}_x \mathbf{q}_1 - 4 \mathbf{d}_z \mathbf{q}_3 + 2 \mathbf{d}_y \mathbf{q}_4 & 2 \mathbf{d}_x \mathbf{q}_2 + 2 \mathbf{d}_y \mathbf{q}_3 \end{pmatrix}^T // \text{MatrixForm}$$

$$\begin{pmatrix} -2 \mathbf{d}_z \mathbf{q}_3 + 2 \mathbf{d}_y \mathbf{q}_4 & 2 \mathbf{d}_z \mathbf{q}_2 - 2 \mathbf{d}_x \mathbf{q}_4 & -2 \mathbf{d}_y \mathbf{q}_2 + 2 \mathbf{d}_x \mathbf{q}_3 \\ 2 \mathbf{d}_y \mathbf{q}_3 + 2 \mathbf{d}_z \mathbf{q}_4 & 2 \mathbf{d}_z \mathbf{q}_1 - 4 \mathbf{d}_y \mathbf{q}_2 + 2 \mathbf{d}_x \mathbf{q}_3 & -2 \mathbf{d}_y \mathbf{q}_1 - 4 \mathbf{d}_z \mathbf{q}_2 + 2 \mathbf{d}_x \mathbf{q}_4 \\ -2 \mathbf{d}_z \mathbf{q}_1 + 2 \mathbf{d}_y \mathbf{q}_2 - 4 \mathbf{d}_x \mathbf{q}_3 & 2 \mathbf{d}_x \mathbf{q}_2 + 2 \mathbf{d}_z \mathbf{q}_4 & 2 \mathbf{d}_x \mathbf{q}_1 - 4 \mathbf{d}_z \mathbf{q}_3 + 2 \mathbf{d}_y \mathbf{q}_4 \\ 2 \mathbf{d}_y \mathbf{q}_1 + 2 \mathbf{d}_z \mathbf{q}_2 - 4 \mathbf{d}_x \mathbf{q}_4 & -2 \mathbf{d}_x \mathbf{q}_1 + 2 \mathbf{d}_z \mathbf{q}_3 - 4 \mathbf{d}_y \mathbf{q}_4 & 2 \mathbf{d}_x \mathbf{q}_2 + 2 \mathbf{d}_y \mathbf{q}_3 \end{pmatrix}$$

The direction of gravity (the vertical axis, z axis)

■ An appropriate convention (the equations simplify)

$${}^E \hat{\mathbf{g}} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

■ Normalized accelerometer measurement

$${}^S \hat{\mathbf{a}} = \begin{pmatrix} 0 & \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \end{pmatrix}$$

■ A objective function

$$\mathbf{f}_g[{}^S \hat{\mathbf{q}}, {}^S \hat{\mathbf{a}}] = \begin{pmatrix} 2 \left(\mathbf{q}_2 \mathbf{q}_4 - \mathbf{q}_1 \mathbf{q}_3 \right) - \mathbf{a}_x \\ 2 \left(\mathbf{q}_1 \mathbf{q}_2 + \mathbf{q}_3 \mathbf{q}_4 \right) - \mathbf{a}_y \\ 2 \left(\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_3^2 \right) - \mathbf{a}_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}_g[{}^S \hat{\mathbf{q}}] = \begin{pmatrix} -2 \mathbf{q}_3 & 2 \mathbf{q}_4 & -2 \mathbf{q}_1 & 2 \mathbf{q}_2 \\ 2 \mathbf{q}_2 & 2 \mathbf{q}_1 & 2 \mathbf{q}_4 & 2 \mathbf{q}_3 \\ 0 & -4 \mathbf{q}_2 & -4 \mathbf{q}_3 & 0 \end{pmatrix}$$

The earth's magnetic field (one horizontal axis, the vertical axis)

■ An appropriate convention (the equations simplify)

$${}^E \hat{\mathbf{b}} = \begin{pmatrix} 0 & \mathbf{b}_x & 0 & \mathbf{b}_z \end{pmatrix}$$

■ Normalized accelerometer measurement

$${}^S \hat{\mathbf{m}} = \begin{pmatrix} 0 & \mathbf{m}_x & \mathbf{m}_y & \mathbf{m}_z \end{pmatrix}$$

■ A objective function

$$\mathbf{f}_b[{}^S \hat{\mathbf{q}}, {}^E \hat{\mathbf{b}}, {}^S \hat{\mathbf{m}}] = \begin{pmatrix} 2 \mathbf{b}_x \left(\frac{1}{2} - \mathbf{q}_3^2 - \mathbf{q}_4^2 \right) + 2 \mathbf{b}_z \left(\mathbf{q}_2 \mathbf{q}_4 - \mathbf{q}_1 \mathbf{q}_3 \right) - \mathbf{m}_x \\ 2 \mathbf{b}_x \left(\mathbf{q}_2 \mathbf{q}_3 - \mathbf{q}_1 \mathbf{q}_4 \right) + 2 \mathbf{b}_z \left(\mathbf{q}_1 \mathbf{q}_2 + \mathbf{q}_3 \mathbf{q}_4 \right) - \mathbf{m}_y \\ 2 \mathbf{b}_x \left(\mathbf{q}_1 \mathbf{q}_3 + \mathbf{q}_2 \mathbf{q}_4 \right) + 2 \mathbf{b}_z \left(\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_3^2 \right) - \mathbf{m}_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}_b[{}^S \hat{\mathbf{q}}, {}^E \hat{\mathbf{b}}] = \begin{pmatrix} -2 \mathbf{b}_z \mathbf{q}_3 & 2 \mathbf{b}_z \mathbf{q}_4 & -4 \mathbf{b}_x \mathbf{q}_3 - 2 \mathbf{b}_z \mathbf{q}_1 & -4 \mathbf{b}_x \mathbf{q}_4 + 2 \mathbf{b}_z \mathbf{q}_2 \\ -2 \mathbf{b}_x \mathbf{q}_4 + 2 \mathbf{b}_z \mathbf{q}_2 & 2 \mathbf{b}_x \mathbf{q}_3 + 2 \mathbf{b}_z \mathbf{q}_1 & 2 \mathbf{b}_x \mathbf{q}_2 + 2 \mathbf{b}_z \mathbf{q}_4 & -2 \mathbf{b}_x \mathbf{q}_1 + 2 \mathbf{b}_z \mathbf{q}_3 \\ 2 \mathbf{b}_x \mathbf{q}_3 & 2 \mathbf{b}_x \mathbf{q}_4 - 4 \mathbf{b}_z \mathbf{q}_2 & 2 \mathbf{b}_x \mathbf{q}_1 - 4 \mathbf{b}_z \mathbf{q}_3 & 2 \mathbf{b}_x \mathbf{q}_2 \end{pmatrix}$$

The solution surface → minimum

■ The objective function

$$\mathbf{f}_{g,b}[{}^S \hat{\mathbf{q}}, {}^S \hat{\mathbf{a}}, {}^E \hat{\mathbf{b}}, {}^S \hat{\mathbf{m}}] = \begin{pmatrix} \mathbf{f}_g[{}^S \hat{\mathbf{q}}, {}^S \hat{\mathbf{a}}] \\ \mathbf{f}_b[{}^S \hat{\mathbf{q}}, {}^E \hat{\mathbf{b}}, {}^S \hat{\mathbf{m}}] \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{pmatrix} = \begin{pmatrix} 2 (\mathbf{q}_2 \mathbf{q}_4 - \mathbf{q}_1 \mathbf{q}_3) - \mathbf{a}_x \\ 2 (\mathbf{q}_1 \mathbf{q}_2 + \mathbf{q}_3 \mathbf{q}_4) - \mathbf{a}_y \\ 2 (\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_3^2) - \mathbf{a}_z \\ 2 \mathbf{b}_x (\frac{1}{2} - \mathbf{q}_3^2 - \mathbf{q}_4^2) + 2 \mathbf{b}_z (\mathbf{q}_2 \mathbf{q}_4 - \mathbf{q}_1 \mathbf{q}_3) - \mathbf{m}_x \\ 2 \mathbf{b}_x (\mathbf{q}_2 \mathbf{q}_3 - \mathbf{q}_1 \mathbf{q}_4) + 2 \mathbf{b}_z (\mathbf{q}_1 \mathbf{q}_2 + \mathbf{q}_3 \mathbf{q}_4) - \mathbf{m}_y \\ 2 \mathbf{b}_x (\mathbf{q}_1 \mathbf{q}_3 + \mathbf{q}_2 \mathbf{q}_4) + 2 \mathbf{b}_z (\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_3^2) - \mathbf{m}_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}_{g,b} [\hat{\mathbf{q}}, \mathbf{b}] = \begin{pmatrix} \mathbf{J}_g^T [\hat{\mathbf{q}}] \\ \mathbf{J}_b^T [\hat{\mathbf{q}}, \mathbf{b}] \end{pmatrix}$$

$$\mathbf{J} = \begin{pmatrix} -2 \mathbf{q}_3 & 2 \mathbf{q}_4 & -2 \mathbf{q}_1 & 2 \mathbf{q}_2 \\ 2 \mathbf{q}_2 & 2 \mathbf{q}_1 & 2 \mathbf{q}_4 & 2 \mathbf{q}_3 \\ 0 & -4 \mathbf{q}_2 & -4 \mathbf{q}_3 & 0 \\ -2 \mathbf{b}_z \mathbf{q}_3 & 2 \mathbf{b}_z \mathbf{q}_4 & -4 \mathbf{b}_x \mathbf{q}_3 - 2 \mathbf{b}_z \mathbf{q}_1 & -4 \mathbf{b}_x \mathbf{q}_4 + 2 \mathbf{b}_z \mathbf{q}_2 \\ -2 \mathbf{b}_x \mathbf{q}_4 + 2 \mathbf{b}_z \mathbf{q}_2 & 2 \mathbf{b}_x \mathbf{q}_3 + 2 \mathbf{b}_z \mathbf{q}_1 & 2 \mathbf{b}_x \mathbf{q}_2 + 2 \mathbf{b}_z \mathbf{q}_4 & -2 \mathbf{b}_x \mathbf{q}_1 + 2 \mathbf{b}_z \mathbf{q}_3 \\ 2 \mathbf{b}_x \mathbf{q}_3 & 2 \mathbf{b}_x \mathbf{q}_4 - 4 \mathbf{b}_z \mathbf{q}_2 & 2 \mathbf{b}_x \mathbf{q}_1 - 4 \mathbf{b}_z \mathbf{q}_3 & 2 \mathbf{b}_x \mathbf{q}_2 \end{pmatrix}$$

The estimate orientation $\hat{\mathbf{q}}_{\nabla, t}$

$$\hat{\mathbf{q}}_{\nabla, t} = \hat{\mathbf{q}}_{\text{est}, t-1} - \mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|}$$

■ The convergence rate governed by $\mu_t \geq$ the physical rate

A previous estimate of orientation $\hat{\mathbf{q}}_{\text{est}, t-1}$

■ The objective function gradient

$$\nabla \mathbf{f} = \begin{Bmatrix} \mathbf{J}_g^T [\hat{\mathbf{q}}_{\text{est}, t-1}] \mathbf{f}_g [\hat{\mathbf{q}}_{\text{est}, t-1}, \mathbf{s}\hat{\mathbf{a}}_t] \\ \mathbf{J}_{g,b}^T [\hat{\mathbf{q}}_{\text{est}, t-1}, \mathbf{b}] \mathbf{f}_{g,b} [\hat{\mathbf{q}}_{\text{est}, t-1}, \mathbf{s}\hat{\mathbf{a}}, \mathbf{b}, \mathbf{s}\hat{\mathbf{m}}] \end{Bmatrix}$$

The convergence rate of $\hat{\mathbf{q}}_{\nabla, t}$

The physical orientation rate $\hat{\mathbf{q}}_{\omega, t}$

■ An optimal value of the step-size

$$\mu_t = \alpha |\hat{\mathbf{q}}_{\omega, t}| \Delta t, \quad \alpha > 1$$

Filter fusion algorithm

■ An estimated orientation

$$\hat{\mathbf{q}}_{\text{est}, t} = \gamma_t \hat{\mathbf{q}}_{\nabla, t} + (1 - \gamma_t) \hat{\mathbf{q}}_{\omega, t}, \quad 0 \leq \gamma_t \leq 1$$

■ $\beta, \frac{\mu_t}{\Delta t}$

$$\text{The weighted divergence of } \hat{\mathbf{q}}_{\omega} \quad \beta \quad \text{The weighted convergence of } \hat{\mathbf{q}}_{\nabla} \quad \frac{\mu_t}{\Delta t}$$

■ The magnitude of a quaternion derivative == The gyroscope measurement error

$$(1 - \gamma_t) \beta = \gamma_t \frac{\mu_t}{\Delta t}$$

$$\gamma_t = \frac{\beta}{\frac{\mu_t}{\Delta t} + \beta}$$

■ The optimal fusion

■ the convergence rate of $\hat{\mathbf{q}}_{\nabla} \geq \alpha \rightarrow$ the physical rate of change of orientation

$$\hat{\mathbf{q}}_{\nabla, t} \approx -\mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|}$$

$$\gamma_t \approx \frac{\beta \Delta t}{\mu_t}$$

■ Simplify

■ An estimated orientation

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \frac{\beta \Delta t}{\mu t} \left(-\mu t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) + (1 - \theta) (\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} \Delta t)$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = -\beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \Delta t + \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} \Delta t$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} \Delta t - \beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \Delta t$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \left(\hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} - \beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \left(\hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} - \beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \left(\hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} - \beta \hat{\mathbf{e}}_{\mathbf{q}_{\epsilon}, t} \right) \Delta t$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} \Delta t$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \left(\frac{1}{2} \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} \otimes \mathbf{s}_{\omega t} - \beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

The estimated orientation rate $\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}}$

The rate of change of orientation measured by the gyroscopes $\hat{\mathbf{e}}_{\mathbf{q}_{\omega}}$

The direction of the estimated error $\hat{\mathbf{e}}_{\mathbf{q}_{\epsilon}}$

■ Simplified to equation

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} \Delta t$$

■ The estimated rate of change of orientation

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} - \beta \hat{\mathbf{e}}_{\mathbf{q}_{\epsilon}, t}$$

■ The quaternion derivative measured at time t

$$\hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} = \frac{1}{2} \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} \otimes \mathbf{s}_{\omega t}$$

■ The direction of the error of $\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t}$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\epsilon}, t} = \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|}$$

■ The objective function

$$\mathbf{f}_{g, b} [\hat{\mathbf{e}}_{\mathbf{q}}, \mathbf{s}_{\mathbf{a}}, \mathbf{e}_{\mathbf{b}}, \mathbf{s}_{\mathbf{m}}] = \begin{pmatrix} \mathbf{f}_g [\hat{\mathbf{e}}_{\mathbf{q}}, \mathbf{s}_{\mathbf{a}}] \\ \mathbf{f}_b [\hat{\mathbf{e}}_{\mathbf{q}}, \mathbf{e}_{\mathbf{b}}, \mathbf{s}_{\mathbf{m}}] \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{pmatrix} = \begin{pmatrix} 2 (\mathbf{q}_2 \mathbf{q}_4 - \mathbf{q}_1 \mathbf{q}_3) - \mathbf{a}_x \\ 2 (\mathbf{q}_1 \mathbf{q}_2 + \mathbf{q}_3 \mathbf{q}_4) - \mathbf{a}_y \\ 2 \left(\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_3^2 \right) - \mathbf{a}_z \\ 2 \mathbf{b}_x \left(\frac{1}{2} - \mathbf{q}_3^2 - \mathbf{q}_4^2 \right) + 2 \mathbf{b}_z (\mathbf{q}_2 \mathbf{q}_4 - \mathbf{q}_1 \mathbf{q}_3) - \mathbf{m}_x \\ 2 \mathbf{b}_x (\mathbf{q}_2 \mathbf{q}_3 - \mathbf{q}_1 \mathbf{q}_4) + 2 \mathbf{b}_z (\mathbf{q}_1 \mathbf{q}_2 + \mathbf{q}_3 \mathbf{q}_4) - \mathbf{m}_y \\ 2 \mathbf{b}_x (\mathbf{q}_1 \mathbf{q}_3 + \mathbf{q}_2 \mathbf{q}_4) + 2 \mathbf{b}_z \left(\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_3^2 \right) - \mathbf{m}_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}_{g, b} [\hat{\mathbf{e}}_{\mathbf{q}}, \mathbf{e}_{\mathbf{b}}] = \begin{pmatrix} \mathbf{J}_g^T [\hat{\mathbf{e}}_{\mathbf{q}}] \\ \mathbf{J}_b^T [\hat{\mathbf{e}}_{\mathbf{q}}, \mathbf{e}_{\mathbf{b}}] \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \mathbf{J}_{13} & \mathbf{J}_{14} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \mathbf{J}_{23} & \mathbf{J}_{24} \\ \mathbf{J}_{31} & \mathbf{J}_{32} & \mathbf{J}_{33} & \mathbf{J}_{34} \\ \mathbf{J}_{41} & \mathbf{J}_{42} & \mathbf{J}_{43} & \mathbf{J}_{44} \\ \mathbf{J}_{51} & \mathbf{J}_{52} & \mathbf{J}_{53} & \mathbf{J}_{54} \\ \mathbf{J}_{61} & \mathbf{J}_{62} & \mathbf{J}_{63} & \mathbf{J}_{64} \end{pmatrix} = \begin{pmatrix} -2 \mathbf{q}_3 & 2 \mathbf{q}_4 & -2 \mathbf{q}_1 & 2 \mathbf{q}_2 \\ 2 \mathbf{q}_2 & 2 \mathbf{q}_1 & 2 \mathbf{q}_4 & 2 \mathbf{q}_3 \\ 0 & -4 \mathbf{q}_2 & -4 \mathbf{q}_3 & 0 \\ -2 \mathbf{b}_z \mathbf{q}_3 & 2 \mathbf{b}_z \mathbf{q}_4 & -4 \mathbf{b}_x \mathbf{q}_3 - 2 \mathbf{b}_z \mathbf{q}_1 & -4 \mathbf{b}_x \mathbf{q}_4 + 2 \mathbf{b}_z \mathbf{q}_2 \\ -2 \mathbf{b}_x \mathbf{q}_4 + 2 \mathbf{b}_z \mathbf{q}_2 & 2 \mathbf{b}_x \mathbf{q}_3 + 2 \mathbf{b}_z \mathbf{q}_1 & 2 \mathbf{b}_x \mathbf{q}_2 + 2 \mathbf{b}_z \mathbf{q}_4 & -2 \mathbf{b}_x \mathbf{q}_1 + 2 \mathbf{b}_z \mathbf{q}_3 \\ 2 \mathbf{b}_x \mathbf{q}_3 & 2 \mathbf{b}_x \mathbf{q}_4 - 4 \mathbf{b}_z \mathbf{q}_2 & 2 \mathbf{b}_x \mathbf{q}_1 - 4 \mathbf{b}_z \mathbf{q}_3 & 2 \mathbf{b}_x \mathbf{q}_2 \end{pmatrix}$$

■ The gradient of the solution surface (General form)

$$\nabla \mathbf{f} = \mathbf{J}^T \mathbf{f}$$

$$\nabla \mathbf{f}_{g,b} = \mathbf{J}_{g,b} \begin{bmatrix} \hat{\mathbf{s}}\mathbf{q}, & {}^E\hat{\mathbf{b}} \end{bmatrix}^T \mathbf{f}_{g,b} \begin{bmatrix} \hat{\mathbf{s}}\mathbf{q}, & {}^S\mathbf{a}, & {}^E\hat{\mathbf{b}}, & {}^S\hat{\mathbf{m}} \end{bmatrix}$$

$$\begin{pmatrix} \nabla \mathbf{f}_1 \\ \nabla \mathbf{f}_2 \\ \nabla \mathbf{f}_3 \\ \nabla \mathbf{f}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \mathbf{J}_{13} & \mathbf{J}_{14} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \mathbf{J}_{23} & \mathbf{J}_{24} \\ \mathbf{J}_{31} & \mathbf{J}_{32} & \mathbf{J}_{33} & \mathbf{J}_{34} \\ \mathbf{J}_{41} & \mathbf{J}_{42} & \mathbf{J}_{43} & \mathbf{J}_{44} \\ \mathbf{J}_{51} & \mathbf{J}_{52} & \mathbf{J}_{53} & \mathbf{J}_{54} \\ \mathbf{J}_{61} & \mathbf{J}_{62} & \mathbf{J}_{63} & \mathbf{J}_{64} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \mathbf{J}_{11} + \mathbf{f}_2 \mathbf{J}_{21} + \mathbf{f}_3 \mathbf{J}_{31} + \mathbf{f}_4 \mathbf{J}_{41} + \mathbf{f}_5 \mathbf{J}_{51} + \mathbf{f}_6 \mathbf{J}_{61} \\ \mathbf{f}_1 \mathbf{J}_{12} + \mathbf{f}_2 \mathbf{J}_{22} + \mathbf{f}_3 \mathbf{J}_{32} + \mathbf{f}_4 \mathbf{J}_{42} + \mathbf{f}_5 \mathbf{J}_{52} + \mathbf{f}_6 \mathbf{J}_{62} \\ \mathbf{f}_1 \mathbf{J}_{13} + \mathbf{f}_2 \mathbf{J}_{23} + \mathbf{f}_3 \mathbf{J}_{33} + \mathbf{f}_4 \mathbf{J}_{43} + \mathbf{f}_5 \mathbf{J}_{53} + \mathbf{f}_6 \mathbf{J}_{63} \\ \mathbf{f}_1 \mathbf{J}_{14} + \mathbf{f}_2 \mathbf{J}_{24} + \mathbf{f}_3 \mathbf{J}_{34} + \mathbf{f}_4 \mathbf{J}_{44} + \mathbf{f}_5 \mathbf{J}_{54} + \mathbf{f}_6 \mathbf{J}_{64} \end{pmatrix}$$

$$\begin{pmatrix} \hat{\mathbf{s}}\dot{\mathbf{q}}_1 \\ \hat{\mathbf{s}}\dot{\mathbf{q}}_2 \\ \hat{\mathbf{s}}\dot{\mathbf{q}}_3 \\ \hat{\mathbf{s}}\dot{\mathbf{q}}_4 \end{pmatrix} = \begin{pmatrix} \frac{\nabla \mathbf{f}_1}{|\nabla \mathbf{f}|} \\ \frac{\nabla \mathbf{f}_2}{|\nabla \mathbf{f}|} \\ \frac{\nabla \mathbf{f}_3}{|\nabla \mathbf{f}|} \\ \frac{\nabla \mathbf{f}_4}{|\nabla \mathbf{f}|} \end{pmatrix}$$

■ Temp

$$\begin{pmatrix} 2 (q_2 q_4 - q_1 q_3) - a_x \\ 2 (q_1 q_2 + q_3 q_4) - a_y \\ 2 \left(\frac{1}{2} - q_2^2 - q_3^2 \right) - a_z \\ 2 b_x \left(\frac{1}{2} - q_3^2 - q_4^2 \right) + 2 b_z (q_2 q_4 - q_1 q_3) - m_x \\ 2 b_x (q_2 q_3 - q_1 q_4) + 2 b_z (q_1 q_2 + q_3 q_4) - m_y \\ 2 b_x (q_1 q_3 + q_2 q_4) + 2 b_z \left(\frac{1}{2} - q_2^2 - q_3^2 \right) - m_z \end{pmatrix}$$

$$\begin{pmatrix} -2 q_3 & 2 q_4 & -2 q_1 & 2 q_2 \\ 2 q_2 & 2 q_1 & 2 q_4 & 2 q_3 \\ 0 & -4 q_2 & -4 q_3 & 0 \end{pmatrix}^T \cdot \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} 2 f_2 q_2 - 2 f_1 q_3 \\ 2 f_2 q_1 - 4 f_3 q_2 + 2 f_1 q_4 \\ -2 f_1 q_1 - 4 f_3 q_3 + 2 f_2 q_4 \\ 2 f_1 q_2 + 2 f_2 q_3 \end{pmatrix}$$

$$\begin{pmatrix} -2 b_z q_3 & 2 b_z q_4 & -4 b_x q_3 - 2 b_z q_1 & -4 b_x q_4 + 2 b_z q_2 \\ -2 b_x q_4 + 2 b_z q_2 & 2 b_x q_3 + 2 b_z q_1 & 2 b_x q_2 + 2 b_z q_4 & -2 b_x q_1 + 2 b_z q_3 \\ 2 b_x q_3 & 2 b_x q_4 - 4 b_z q_2 & 2 b_x q_1 - 4 b_z q_3 & 2 b_x q_2 \end{pmatrix}^T \cdot \begin{pmatrix} \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} -2 b_z f_4 q_3 + 2 b_x f_6 q_3 + f_5 (2 b_z q_2 - 2 b_x q_4) \\ f_5 (2 b_z q_1 + 2 b_x q_3) + 2 b_z f_4 q_4 + f_6 (-4 b_z q_2 + 2 b_x q_4) \\ f_4 (-2 b_z q_1 - 4 b_x q_3) + f_6 (2 b_x q_1 - 4 b_z q_3) + f_5 (2 b_x q_2 + 2 b_z q_4) \\ 2 b_x f_6 q_2 + f_5 (-2 b_x q_1 + 2 b_z q_3) + f_4 (2 b_z q_2 - 4 b_x q_4) \end{pmatrix}$$

Gyroscope bias drift compensation

■ With temperature and with motion

Karman - based approaches → estimate the gyroscope bias

Mahony et al → gyroscope bias drift (the integral feedback of the error)

The normalized direction of the estimated error in the rate of change of orientation $\hat{\mathbf{s}}\dot{\mathbf{q}}_e$

■ The gyroscope bias ${}^S\omega_b$

$${}^S\omega_{b,t} = \zeta \sum_t {}^S\omega_{e,t} \Delta t$$

The integral gain ζ

■ DC component of ${}^S\omega_e$

$${}^S\omega_{e,t} = 2 \hat{\mathbf{s}}\hat{\mathbf{q}}_{\text{est},t-1}^* \otimes \hat{\mathbf{s}}\dot{\mathbf{q}}_{e,t}$$

$$\begin{pmatrix} {}^S\omega_{ex} \\ {}^S\omega_{ey} \\ {}^S\omega_{ez} \end{pmatrix} = 2 \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} \otimes \begin{pmatrix} \hat{\mathbf{s}}\dot{\mathbf{q}}_1 & \hat{\mathbf{s}}\dot{\mathbf{q}}_2 & \hat{\mathbf{s}}\dot{\mathbf{q}}_3 & \hat{\mathbf{s}}\dot{\mathbf{q}}_4 \end{pmatrix} = 2 \begin{pmatrix} q_1 * \hat{\mathbf{s}}\dot{\mathbf{q}}_2 - q_2 * \hat{\mathbf{s}}\dot{\mathbf{q}}_1 - q_3 * \hat{\mathbf{s}}\dot{\mathbf{q}}_4 + q_4 * \hat{\mathbf{s}}\dot{\mathbf{q}}_3 \\ q_1 * \hat{\mathbf{s}}\dot{\mathbf{q}}_3 + q_2 * \hat{\mathbf{s}}\dot{\mathbf{q}}_4 - q_3 * \hat{\mathbf{s}}\dot{\mathbf{q}}_1 - q_4 * \hat{\mathbf{s}}\dot{\mathbf{q}}_2 \\ q_1 * \hat{\mathbf{s}}\dot{\mathbf{q}}_4 - q_2 * \hat{\mathbf{s}}\dot{\mathbf{q}}_3 + q_3 * \hat{\mathbf{s}}\dot{\mathbf{q}}_2 - q_4 * \hat{\mathbf{s}}\dot{\mathbf{q}}_1 \end{pmatrix}$$

■ Gyroscope measurements ${}^S\omega_c$

$${}^S\omega_{c,t} = {}^S\omega_{t} - {}^S\omega_{b,t}$$

$$\begin{pmatrix} w_{bx} \\ w_{by} \\ w_{bz} \end{pmatrix} = \begin{pmatrix} w_{bx} \\ w_{by} \\ w_{bz} \end{pmatrix} + \begin{pmatrix} w_{ex} \\ w_{ey} \\ w_{ez} \end{pmatrix} * \Delta t * \zeta$$

$$\begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} - \begin{pmatrix} w_{bx} \\ w_{by} \\ w_{bz} \end{pmatrix}$$

Filter gains

The estimated mean zero gyroscope measurement error of each axis $\tilde{\omega}_\beta$

- The filter gain

$$\beta = \left| \frac{1}{2} \mathbf{q} \otimes \left(\mathbf{0} \quad \tilde{\omega}_\beta \quad \tilde{\omega}_\beta \quad \tilde{\omega}_\beta \right) \right| = \sqrt{\frac{3}{4}} \tilde{\omega}_\beta$$

The estimated rate of gyroscope bias drift in each axis $\dot{\omega}_\xi$

- The filter gain

$$\xi = \xi \sqrt{\frac{3}{4}} \tilde{\omega}_\xi$$

Orientation from angular rate

- A tri-axis gyroscope

- The angular rate

$$\mathbf{s}_\omega = \left(\mathbf{0} \quad \omega_x \quad \omega_y \quad \omega_z \right)$$

- The quaternion derivative

$$\mathbf{s}_\mathbf{q} = \frac{1}{2} \mathbf{s}_\mathbf{q} \otimes \mathbf{s}_\omega$$

- The quaternion derivative measured at time t

$$\mathbf{s}_\mathbf{q}_{\omega, t} = \frac{1}{2} \mathbf{s}_\mathbf{q}_{\text{est}, t-1} \otimes \mathbf{s}_{\omega t}$$

$$\left(\mathbf{q}_{\omega 1} \quad \mathbf{q}_{\omega 2} \quad \mathbf{q}_{\omega 3} \quad \mathbf{q}_{\omega 4} \right) = \frac{1}{2} \left(\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \quad \mathbf{q}_4 \right) \otimes \left(\mathbf{0} \quad \omega_x \quad \omega_y \quad \omega_z \right)$$

- The orientation of the earth frame at time t

$$\mathbf{s}_\mathbf{q}_{\omega, t} = \mathbf{s}_\mathbf{q}_{\text{est}, t-1} + \mathbf{s}_\mathbf{q}_{\omega, t} \Delta t$$

The samling period Δt

$$\left(\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \quad \mathbf{q}_4 \right) = \left(\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \quad \mathbf{q}_4 \right) + \left(\left(\mathbf{q}_{\omega 1} \quad \mathbf{q}_{\omega 2} \quad \mathbf{q}_{\omega 3} \quad \mathbf{q}_{\omega 4} \right) - \beta \left(\mathbf{s}_\mathbf{q}_1 \quad \mathbf{s}_\mathbf{q}_2 \quad \mathbf{s}_\mathbf{q}_3 \quad \mathbf{s}_\mathbf{q}_4 \right) \right) * \Delta t$$

Magnetic distortion compensation

- Declination errors

The horizontal plane : the earth ' s surface (Heading)

- Inclination errors

The vertical plane : the earth ' s surface (Sensor ' s attitude)

- The measured direction of the earth's magnetic field in the earth frame at time t

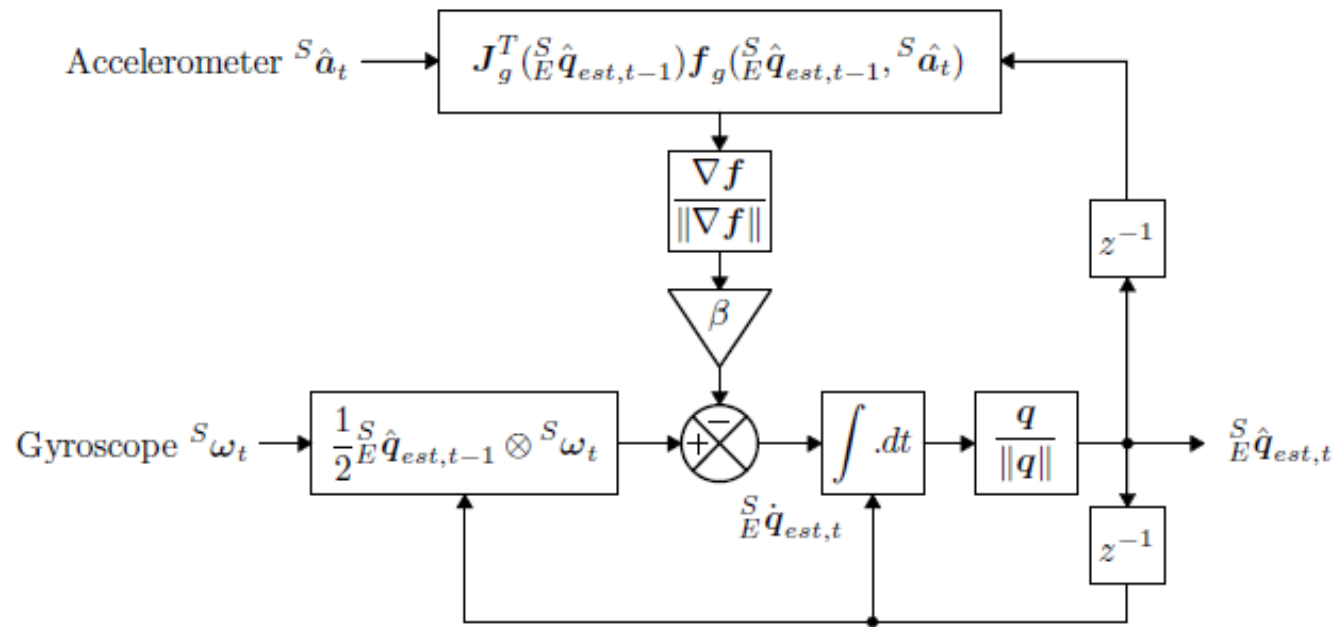
$$\mathbf{e}\hat{\mathbf{h}}_t = \left(\mathbf{0} \quad h_x \quad h_y \quad h_z \right) = \mathbf{s}_\mathbf{q}_{\text{est}, t-1} \otimes \mathbf{s}_\mathbf{m}_t \otimes \mathbf{s}_\mathbf{q}_{\text{est}, t-1}^*$$

$$\begin{aligned}
\begin{pmatrix} 0 & h_x & h_y & h_z \end{pmatrix} &= {}^S\hat{q}_{est,t-1} \otimes {}^S\hat{m}_t \otimes {}^S\hat{q}_{est,t-1}^* \\
&= \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & m_x & m_y & m_z \end{pmatrix} \otimes \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \end{pmatrix} \\
&= \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} \cdot \begin{pmatrix} 0 & m_x & m_y & m_z \\ -m_x & 0 & -m_z & m_y \\ -m_y & m_z & 0 & -m_x \\ -m_z & -m_y & m_x & 0 \end{pmatrix} \cdot \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & q_4 & -q_3 \\ q_3 & -q_4 & q_1 & q_2 \\ q_4 & q_3 & -q_2 & q_1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ m_x (q_1^2 + q_2^2 - q_3^2 - q_4^2) + 2 (m_y (q_2 q_3 - q_1 q_4) + m_z (q_1 q_3 + q_2 q_4)) \\ m_y (q_1^2 - q_2^2 + q_3^2 - q_4^2) + 2 (m_x (q_2 q_3 + q_1 q_4) + m_z (-q_1 q_2 + q_3 q_4)) \\ m_z (q_1^2 - q_2^2 - q_3^2 + q_4^2) + 2 (m_x (-q_1 q_3 + q_2 q_4) + m_y (q_1 q_2 + q_3 q_4)) \end{pmatrix}^T \\
&= \begin{pmatrix} 0 \\ 2 (m_x (\frac{1}{2} - q_3^2 - q_4^2) + m_y (q_2 q_3 - q_1 q_4) + m_z (q_1 q_3 + q_2 q_4)) \\ 2 (m_x (q_2 q_3 + q_1 q_4) + m_y (\frac{1}{2} - q_2^2 - q_4^2) + m_z (-q_1 q_2 + q_3 q_4)) \\ 2 (m_x (-q_1 q_3 + q_2 q_4) + m_y (q_1 q_2 + q_3 q_4) + m_z (\frac{1}{2} - q_2^2 - q_3^2)) \end{pmatrix}^T \\
{}^E\hat{b}_t &= \begin{pmatrix} 0 & \sqrt{h_x^2 + h_y^2} & 0 & h_z \end{pmatrix} \\
\begin{pmatrix} 0 & b_x & b_y & b_z \end{pmatrix} &= \begin{pmatrix} 0 & \sqrt{h_x^2 + h_y^2} & 0 & h_z \end{pmatrix}
\end{aligned}$$

Block diagram

■ IMU

the complete orientation filter for an IMU implementation

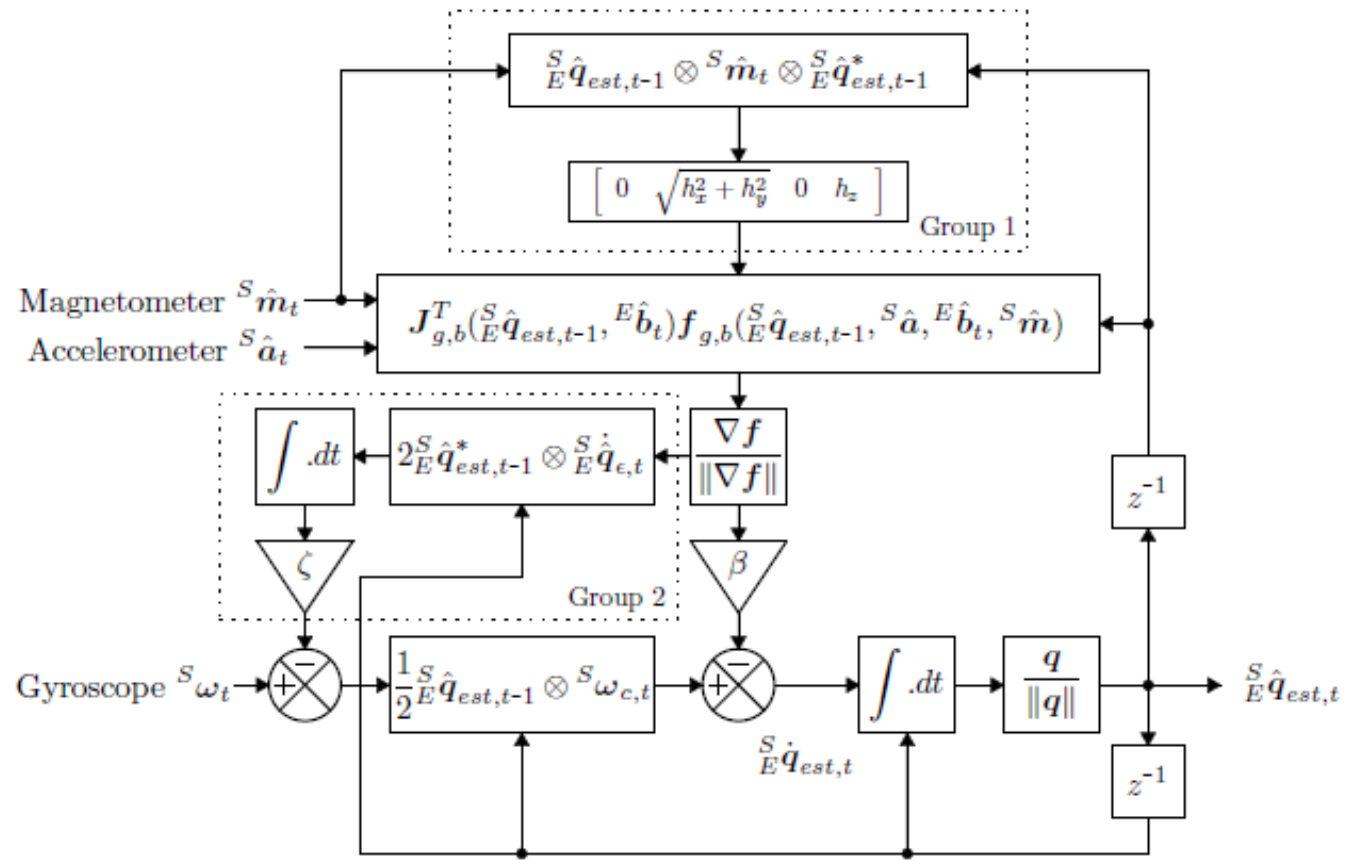


■ MARG

the complete orientation filter for an MARG implementation
including magnetic distortion (Group 1) and gyroscope drift (Group 2) compensation

Group 1 : Magnetic distortion

Group 2 : Gyroscope drift



Filter Readjustment

Orientation from vector observations

$${}^{\mathbb{S}}\hat{\mathbf{q}} = (q_1 \quad q_2 \quad q_3 \quad q_4)$$

$${}^{\mathbb{E}}\hat{\mathbf{d}} = (0 \quad d_x \quad d_y \quad d_z)$$

$${}^{\mathbb{S}}\hat{\mathbf{s}} = (0 \quad s_x \quad s_y \quad s_z)$$

$$\mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}] = {}^{\mathbb{S}}\hat{\mathbf{q}}^* \otimes {}^{\mathbb{E}}\hat{\mathbf{d}} \otimes {}^{\mathbb{S}}\hat{\mathbf{q}} - {}^{\mathbb{S}}\hat{\mathbf{s}}$$

$$\min_{{}^{\mathbb{S}}\hat{\mathbf{q}} \in \mathbb{R}^4} \mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}]$$

$${}^{\mathbb{S}}\hat{\mathbf{q}}_{k+1} = {}^{\mathbb{S}}\hat{\mathbf{q}}_k - \mu \frac{\nabla \mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}]}{|\nabla \mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}|}, \quad k = 0, 1, \dots, n$$

$$\nabla \mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}_k, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}] = \mathbf{J}^{\top}[{}^{\mathbb{S}}\hat{\mathbf{q}}_k, {}^{\mathbb{E}}\hat{\mathbf{d}}] \mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}_k, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}]$$

■ A objective function

$${}^{\mathbb{S}}\hat{\mathbf{q}}^* \otimes {}^{\mathbb{E}}\hat{\mathbf{d}} \otimes {}^{\mathbb{S}}\hat{\mathbf{q}} = (q_1 \quad -q_2 \quad -q_3 \quad -q_4) \otimes (0 \quad d_x \quad d_y \quad d_z) \otimes (q_1 \quad q_2 \quad q_3 \quad q_4)$$

$$= (q_1 \quad -q_2 \quad -q_3 \quad -q_4) \cdot \begin{pmatrix} 0 & d_x & d_y & d_z \\ -d_x & 0 & -d_z & d_y \\ -d_y & d_z & 0 & -d_x \\ -d_z & -d_y & d_x & 0 \end{pmatrix} \cdot \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \\ -q_2 & q_1 & -q_4 & q_3 \\ -q_3 & q_4 & q_1 & -q_2 \\ -q_4 & -q_3 & q_2 & q_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2(d_x(\frac{1}{2} - q_3^2 - q_4^2) + d_y(q_2 q_3 + q_1 q_4) + d_z(q_2 q_4 - q_1 q_3)) \\ 2(d_x(q_2 q_3 - q_1 q_4) + d_y(\frac{1}{2} - q_2^2 - q_4^2) + d_z(q_3 q_4 + q_1 q_2)) \\ 2(d_x(q_2 q_4 + q_1 q_3) + d_y(q_3 q_4 - q_1 q_2) + d_z(\frac{1}{2} - q_2^2 - q_3^2)) \end{pmatrix}^{\top}$$

$$\mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}_k, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}] = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 2 d_x (\frac{1}{2} - q_3^2 - q_4^2) + 2 d_y (q_2 q_3 + q_1 q_4) + 2 d_z (q_2 q_4 - q_1 q_3) - s_x \\ 2 d_x (q_2 q_3 - q_1 q_4) + 2 d_y (\frac{1}{2} - q_2^2 - q_4^2) + 2 d_z (q_3 q_4 + q_1 q_2) - s_y \\ 2 d_x (q_2 q_4 + q_1 q_3) + 2 d_y (q_3 q_4 - q_1 q_2) + 2 d_z (\frac{1}{2} - q_2^2 - q_3^2) - s_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}[{}^{\mathbb{S}}\hat{\mathbf{q}}_k, {}^{\mathbb{E}}\hat{\mathbf{d}}] = \begin{pmatrix} \partial_{q_1} f_1 & \partial_{q_2} f_1 & \partial_{q_3} f_1 & \partial_{q_4} f_1 \\ \partial_{q_1} f_2 & \partial_{q_2} f_2 & \partial_{q_3} f_2 & \partial_{q_4} f_2 \\ \partial_{q_1} f_3 & \partial_{q_2} f_3 & \partial_{q_3} f_3 & \partial_{q_4} f_3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 d_y q_4 - 2 d_z q_3 & 2 d_y q_3 + 2 d_z q_4 & -4 d_x q_3 + 2 d_y q_2 - 2 d_z q_1 & -4 d_x q_4 + 2 d_y q_1 + 2 d_z q_2 \\ -2 d_x q_4 + 2 d_z q_2 & 2 d_x q_3 - 4 d_y q_2 + 2 d_z q_1 & 2 d_x q_2 + 2 d_z q_4 & -2 d_x q_1 - 4 d_y q_4 + 2 d_z q_3 \\ 2 d_x q_3 - 2 d_y q_2 & 2 d_x q_4 - 2 d_y q_1 - 4 d_z q_2 & 2 d_x q_1 + 2 d_y q_4 - 4 d_z q_3 & 2 d_x q_2 + 2 d_y q_3 \end{pmatrix}$$

The direction of gravity (the vertical axis, z axis)

$${}^{\mathbb{E}}\hat{\mathbf{g}} = (0 \quad 0 \quad 0 \quad g_z)$$

$${}^{\mathbb{S}}\hat{\mathbf{a}} = (0 \quad a_x \quad a_y \quad a_z)$$

$$\mathbf{f}_g[{}^{\mathbb{S}}\hat{\mathbf{q}}, {}^{\mathbb{S}}\hat{\mathbf{a}}] = \begin{pmatrix} 2 g_z (q_2 q_4 - q_1 q_3) - a_x \\ 2 g_z (q_1 q_2 + q_3 q_4) - a_y \\ 2 g_z (\frac{1}{2} - q_2^2 - q_3^2) - a_z \end{pmatrix}$$

$$\mathbf{J}_g[{}^{\mathbb{S}}\hat{\mathbf{q}}] = \begin{pmatrix} -2 q_3 & 2 q_4 & -2 q_1 & 2 q_2 \\ 2 q_2 & 2 q_1 & 2 q_4 & 2 q_3 \\ 0 & -4 q_2 & -4 q_3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \mathbf{J}_{13} & \mathbf{J}_{14} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \mathbf{J}_{23} & \mathbf{J}_{24} \\ \mathbf{J}_{31} & \mathbf{J}_{32} & \mathbf{J}_{33} & \mathbf{J}_{34} \end{pmatrix} = \begin{pmatrix} -2 q_3 & 2 q_4 & -2 q_1 & 2 q_2 \\ 2 q_2 & 2 q_1 & 2 q_4 & 2 q_3 \\ 0 & -4 q_2 & -4 q_3 & 0 \end{pmatrix};$$

$$\begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \mathbf{J}_{13} & \mathbf{J}_{14} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \mathbf{J}_{23} & \mathbf{J}_{24} \\ \mathbf{J}_{31} & \mathbf{J}_{32} & \mathbf{J}_{33} & \mathbf{J}_{34} \end{pmatrix}^{\top} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} // \text{FullSimplify} // \text{MatrixForm}$$

$$\begin{pmatrix} 2 f_2 q_2 - 2 f_1 q_3 \\ 2 (f_2 q_1 - 2 f_3 q_2 + f_1 q_4) \\ -2 f_1 q_1 - 4 f_3 q_3 + 2 f_2 q_4 \\ 2 (f_1 q_2 + f_2 q_3) \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{pmatrix} = \begin{pmatrix} 2 g_z (q_2 q_4 - q_1 q_3) - a_x \\ 2 g_z (q_1 q_2 + q_3 q_4) - a_y \\ 2 g_z (\frac{1}{2} - q_2^2 - q_3^2) - a_z \end{pmatrix};$$

$$\begin{pmatrix} \partial_{q_1} \mathbf{f}_1 & \partial_{q_2} \mathbf{f}_1 & \partial_{q_3} \mathbf{f}_1 & \partial_{q_4} \mathbf{f}_1 \\ \partial_{q_1} \mathbf{f}_2 & \partial_{q_2} \mathbf{f}_2 & \partial_{q_3} \mathbf{f}_2 & \partial_{q_4} \mathbf{f}_2 \\ \partial_{q_1} \mathbf{f}_3 & \partial_{q_2} \mathbf{f}_3 & \partial_{q_3} \mathbf{f}_3 & \partial_{q_4} \mathbf{f}_3 \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} -2 g_z q_3 & 2 g_z q_4 & -2 g_z q_1 & 2 g_z q_2 \\ 2 g_z q_2 & 2 g_z q_1 & 2 g_z q_4 & 2 g_z q_3 \\ 0 & -4 g_z q_2 & -4 g_z q_3 & 0 \end{pmatrix}$$

The earth's magnetic field (one horizontal axis, the vertical axis)

$${}^E \hat{\mathbf{b}} = (\mathbf{0} \quad b_x \quad \mathbf{0} \quad b_z)$$

$${}^S \hat{\mathbf{m}} = (\mathbf{0} \quad m_x \quad m_y \quad m_z)$$

$$\mathbf{f}_b[{}^S \hat{\mathbf{q}}, {}^E \hat{\mathbf{b}}, {}^S \hat{\mathbf{m}}] = \begin{pmatrix} 2 b_x (\frac{1}{2} - q_3^2 - q_4^2) + 2 b_z (q_2 q_4 - q_1 q_3) - m_x \\ 2 b_x (q_2 q_3 - q_1 q_4) + 2 b_z (q_1 q_2 + q_3 q_4) - m_y \\ 2 b_x (q_1 q_3 + q_2 q_4) + 2 b_z (\frac{1}{2} - q_2^2 - q_3^2) - m_z \end{pmatrix}$$

$$\mathbf{J}_b[{}^S \hat{\mathbf{q}}, {}^E \hat{\mathbf{b}}] = \begin{pmatrix} -2 b_x q_3 & 2 b_z q_4 & -4 b_x q_3 - 2 b_z q_1 & -4 b_x q_4 + 2 b_z q_2 \\ -2 b_x q_4 + 2 b_z q_2 & 2 b_x q_3 + 2 b_z q_1 & 2 b_x q_2 + 2 b_z q_4 & -2 b_x q_1 + 2 b_z q_3 \\ 2 b_x q_3 & 2 b_x q_4 - 4 b_z q_2 & 2 b_x q_1 - 4 b_z q_3 & 2 b_x q_2 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \mathbf{J}_{13} & \mathbf{J}_{14} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \mathbf{J}_{23} & \mathbf{J}_{24} \\ \mathbf{J}_{31} & \mathbf{J}_{32} & \mathbf{J}_{33} & \mathbf{J}_{34} \end{pmatrix} = \begin{pmatrix} -2 b_x q_3 & 2 b_z q_4 & -4 b_x q_3 - 2 b_z q_1 & -4 b_x q_4 + 2 b_z q_2 \\ -2 b_x q_4 + 2 b_z q_2 & 2 b_x q_3 + 2 b_z q_1 & 2 b_x q_2 + 2 b_z q_4 & -2 b_x q_1 + 2 b_z q_3 \\ 2 b_x q_3 & 2 b_x q_4 - 4 b_z q_2 & 2 b_x q_1 - 4 b_z q_3 & 2 b_x q_2 \end{pmatrix};$$

$$\begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \mathbf{J}_{13} & \mathbf{J}_{14} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \mathbf{J}_{23} & \mathbf{J}_{24} \\ \mathbf{J}_{31} & \mathbf{J}_{32} & \mathbf{J}_{33} & \mathbf{J}_{34} \end{pmatrix}^T \cdot \begin{pmatrix} \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{pmatrix} // \text{FullSimplify} // \text{MatrixForm}$$

$$\begin{pmatrix} 2 (f_5 h_z q_2 + b_x f_6 q_3 - f_4 h_z q_3 - b_x f_5 q_4) \\ 2 (f_5 h_z q_1 - 2 f_6 h_z q_2 + b_x f_5 q_3 + b_x f_6 q_4 + f_4 h_z q_4) \\ 2 (b_x (f_6 q_1 + f_5 q_2 - 2 f_4 q_3) + h_z (-f_4 q_1 - 2 f_6 q_3 + f_5 q_4)) \\ 2 (-b_x f_5 q_1 + b_x f_6 q_2 + f_4 h_z q_2 + f_5 h_z q_3 - 2 b_x f_4 q_4) \end{pmatrix}$$

The gradient of the solution surface (General form)

$$\nabla \mathbf{f}[{}^S \hat{\mathbf{q}}_k, {}^E \hat{\mathbf{a}}, {}^S \hat{\mathbf{s}}] = \mathbf{J}[{}^S \hat{\mathbf{q}}_k, {}^E \hat{\mathbf{a}}]^T \cdot \mathbf{f}[{}^S \hat{\mathbf{q}}_k, {}^E \hat{\mathbf{a}}, {}^S \hat{\mathbf{s}}]$$

■ The objective function

$$\mathbf{f}_{g,b}[{}^S \hat{\mathbf{q}}, {}^S \hat{\mathbf{a}}, {}^E \hat{\mathbf{b}}, {}^S \hat{\mathbf{m}}] = \begin{pmatrix} \mathbf{f}_g[{}^S \hat{\mathbf{q}}, {}^S \hat{\mathbf{a}}] \\ \mathbf{f}_b[{}^S \hat{\mathbf{q}}, {}^E \hat{\mathbf{b}}, {}^S \hat{\mathbf{m}}] \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{pmatrix} = \begin{pmatrix} 2 (q_2 q_4 - q_1 q_3) - a_x \\ 2 (q_1 q_2 + q_3 q_4) - a_y \\ 2 (\frac{1}{2} - q_2^2 - q_3^2) - a_z \\ 2 b_x (\frac{1}{2} - q_3^2 - q_4^2) + 2 b_z (q_2 q_4 - q_1 q_3) - m_x \\ 2 b_x (q_2 q_3 - q_1 q_4) + 2 b_z (q_1 q_2 + q_3 q_4) - m_y \\ 2 b_x (q_1 q_3 + q_2 q_4) + 2 b_z (\frac{1}{2} - q_2^2 - q_3^2) - m_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}_{g,b}[{}^S \hat{\mathbf{q}}, {}^E \hat{\mathbf{b}}] = \begin{pmatrix} \mathbf{J}_g^T[{}^S \hat{\mathbf{q}}] \\ \mathbf{J}_b^T[{}^S \hat{\mathbf{q}}, {}^E \hat{\mathbf{b}}] \end{pmatrix}$$

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \mathbf{J}_{13} & \mathbf{J}_{14} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \mathbf{J}_{23} & \mathbf{J}_{24} \\ \mathbf{J}_{31} & \mathbf{J}_{32} & \mathbf{J}_{33} & \mathbf{J}_{34} \\ \mathbf{J}_{41} & \mathbf{J}_{42} & \mathbf{J}_{43} & \mathbf{J}_{44} \\ \mathbf{J}_{51} & \mathbf{J}_{52} & \mathbf{J}_{53} & \mathbf{J}_{54} \\ \mathbf{J}_{61} & \mathbf{J}_{62} & \mathbf{J}_{63} & \mathbf{J}_{64} \end{pmatrix} = \begin{pmatrix} -2 q_3 & 2 q_4 & -2 q_1 & 2 q_2 \\ 2 q_2 & 2 q_1 & 2 q_4 & 2 q_3 \\ 0 & -4 q_2 & -4 q_3 & 0 \\ -2 b_x q_3 & 2 b_z q_4 & -4 b_x q_3 - 2 b_z q_1 & -4 b_x q_4 + 2 b_z q_2 \\ -2 b_x q_4 + 2 b_z q_2 & 2 b_x q_3 + 2 b_z q_1 & 2 b_x q_2 + 2 b_z q_4 & -2 b_x q_1 + 2 b_z q_3 \\ 2 b_x q_3 & 2 b_x q_4 - 4 b_z q_2 & 2 b_x q_1 - 4 b_z q_3 & 2 b_x q_2 \end{pmatrix}$$

■ gradient

$$\nabla \mathbf{f}_{g,b} = \mathbf{J}_{g,b}[{}^S \hat{\mathbf{q}}, {}^E \hat{\mathbf{b}}]^T \cdot \mathbf{f}_{g,b}[{}^S \hat{\mathbf{q}}, {}^S \hat{\mathbf{a}}, {}^E \hat{\mathbf{b}}, {}^S \hat{\mathbf{m}}]$$

$$\begin{pmatrix} \nabla \mathbf{f}_1 \\ \nabla \mathbf{f}_2 \\ \nabla \mathbf{f}_3 \\ \nabla \mathbf{f}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \mathbf{J}_{13} & \mathbf{J}_{14} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \mathbf{J}_{23} & \mathbf{J}_{24} \\ \mathbf{J}_{31} & \mathbf{J}_{32} & \mathbf{J}_{33} & \mathbf{J}_{34} \\ \mathbf{J}_{41} & \mathbf{J}_{42} & \mathbf{J}_{43} & \mathbf{J}_{44} \\ \mathbf{J}_{51} & \mathbf{J}_{52} & \mathbf{J}_{53} & \mathbf{J}_{54} \\ \mathbf{J}_{61} & \mathbf{J}_{62} & \mathbf{J}_{63} & \mathbf{J}_{64} \end{pmatrix}^T \cdot \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \mathbf{J}_{11} + \mathbf{f}_2 \mathbf{J}_{21} + \mathbf{f}_3 \mathbf{J}_{31} + \mathbf{f}_4 \mathbf{J}_{41} + \mathbf{f}_5 \mathbf{J}_{51} + \mathbf{f}_6 \mathbf{J}_{61} \\ \mathbf{f}_1 \mathbf{J}_{12} + \mathbf{f}_2 \mathbf{J}_{22} + \mathbf{f}_3 \mathbf{J}_{32} + \mathbf{f}_4 \mathbf{J}_{42} + \mathbf{f}_5 \mathbf{J}_{52} + \mathbf{f}_6 \mathbf{J}_{62} \\ \mathbf{f}_1 \mathbf{J}_{13} + \mathbf{f}_2 \mathbf{J}_{23} + \mathbf{f}_3 \mathbf{J}_{33} + \mathbf{f}_4 \mathbf{J}_{43} + \mathbf{f}_5 \mathbf{J}_{53} + \mathbf{f}_6 \mathbf{J}_{63} \\ \mathbf{f}_1 \mathbf{J}_{14} + \mathbf{f}_2 \mathbf{J}_{24} + \mathbf{f}_3 \mathbf{J}_{34} + \mathbf{f}_4 \mathbf{J}_{44} + \mathbf{f}_5 \mathbf{J}_{54} + \mathbf{f}_6 \mathbf{J}_{64} \end{pmatrix}$$

$$\begin{pmatrix} \hat{\mathbf{q}}_1 \\ \hat{\mathbf{q}}_2 \\ \hat{\mathbf{q}}_3 \\ \hat{\mathbf{q}}_4 \end{pmatrix} = \begin{pmatrix} \frac{\nabla f_1}{|\nabla \mathbf{f}|} \\ \frac{\nabla f_2}{|\nabla \mathbf{f}|} \\ \frac{\nabla f_3}{|\nabla \mathbf{f}|} \\ \frac{\nabla f_4}{|\nabla \mathbf{f}|} \end{pmatrix}$$

The estimate orientation $\hat{\mathbf{q}}_{\nabla, t}$

$$\hat{\mathbf{q}}_{\nabla, t} = \hat{\mathbf{q}}_{\text{est}, t-1} - \mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|}$$

$$\mu_t = \alpha \|\hat{\mathbf{q}}_{\omega, t}\| \Delta t, \alpha > 1$$

Orientation from angular rate

$$\mathbf{S}_{\omega} = \begin{pmatrix} 0 & \omega_x & \omega_y & \omega_z \end{pmatrix}$$

$$\hat{\mathbf{q}}_{\omega, t} = \frac{1}{2} \hat{\mathbf{q}}_{\text{est}, t-1} \otimes \mathbf{S}_{\omega t}$$

$$\begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & \omega_x & \omega_y & \omega_z \end{pmatrix}$$

$$\hat{\mathbf{q}}_{\omega, t} = \hat{\mathbf{q}}_{\text{est}, t-1} + \hat{\mathbf{q}}_{\omega, t} \Delta t$$

Magnetic distortion compensation

$$\begin{aligned} \mathbf{F} \hat{\mathbf{h}}_t &= \begin{pmatrix} 0 & h_x & h_y & h_z \end{pmatrix} = \hat{\mathbf{q}}_{\text{est}, t-1} \otimes \mathbf{S} \hat{\mathbf{m}}_t \otimes \hat{\mathbf{q}}_{\text{est}, t-1}^* \\ &= \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & m_x & m_y & m_z \end{pmatrix} \otimes \begin{pmatrix} \mathbf{q}_1 & -\mathbf{q}_2 & -\mathbf{q}_3 & -\mathbf{q}_4 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 \end{pmatrix} \cdot \begin{pmatrix} 0 & m_x & m_y & m_z \\ -m_x & 0 & -m_z & m_y \\ -m_y & m_z & 0 & -m_x \\ -m_z & -m_y & m_x & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{q}_1 & -\mathbf{q}_2 & -\mathbf{q}_3 & -\mathbf{q}_4 \\ \mathbf{q}_2 & \mathbf{q}_1 & \mathbf{q}_4 & -\mathbf{q}_3 \\ \mathbf{q}_3 & -\mathbf{q}_4 & \mathbf{q}_1 & \mathbf{q}_2 \\ \mathbf{q}_4 & \mathbf{q}_3 & -\mathbf{q}_2 & \mathbf{q}_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 2(m_x(\frac{1}{2} - q_3^2 - q_4^2) + m_y(q_2 q_3 - q_1 q_4) + m_z(q_1 q_3 + q_2 q_4)) \\ 2(m_x(q_2 q_3 + q_1 q_4) + m_y(\frac{1}{2} - q_2^2 - q_4^2) + m_z(-q_1 q_2 + q_3 q_4)) \\ 2(m_x(-q_1 q_3 + q_2 q_4) + m_y(q_1 q_2 + q_3 q_4) + m_z(\frac{1}{2} - q_2^2 - q_3^2)) \end{pmatrix}^T \\ \mathbf{F} \hat{\mathbf{b}}_t &= \begin{pmatrix} 0 & \sqrt{h_x^2 + h_y^2} & 0 & h_z \end{pmatrix} \\ \begin{pmatrix} 0 & b_x & b_y & b_z \end{pmatrix} &= \begin{pmatrix} 0 & \sqrt{h_x^2 + h_y^2} & 0 & h_z \end{pmatrix} \end{aligned}$$

Filter fusion algorithm

■ $(1 - \gamma_t) \beta = \gamma_t \frac{\mu_t}{\Delta t}$

■ $\hat{\mathbf{q}}_{\omega, t} = \hat{\mathbf{q}}_{\text{est}, t-1} + \hat{\mathbf{q}}_{\omega, t} \Delta t$

$$\hat{\mathbf{q}}_{\omega, t} = \frac{1}{2} \hat{\mathbf{q}}_{\text{est}, t-1} \otimes \mathbf{S}_{\omega t}$$

■ $\hat{\mathbf{q}}_{\nabla, t} = \hat{\mathbf{q}}_{\text{est}, t-1} - \mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|}$

$$\mu_t = \alpha \|\hat{\mathbf{q}}_{\omega, t}\| \Delta t, \alpha > 1$$

■ $\hat{\mathbf{q}}_{\text{est}, t} = \gamma_t \hat{\mathbf{q}}_{\nabla, t} + (1 - \gamma_t) \hat{\mathbf{q}}_{\omega, t}, \quad 0 \leq \gamma_t \leq 1$

$$\hat{\mathbf{q}}_{\text{est}, t} = \gamma_t \left(\hat{\mathbf{q}}_{\text{est}, t-1} - \mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) + (1 - \gamma_t) (\hat{\mathbf{q}}_{\text{est}, t-1} + \hat{\mathbf{q}}_{\omega, t} \Delta t)$$

$$\hat{\mathbf{q}}_{\text{est}, t} = \hat{\mathbf{q}}_{\text{est}, t-1} - \gamma_t \mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} + (1 - \gamma_t) \hat{\mathbf{q}}_{\omega, t} \Delta t$$

$$\hat{\mathbf{q}}_{\text{est}, t} = \hat{\mathbf{q}}_{\text{est}, t-1} + \left((1 - \gamma_t) \hat{\mathbf{q}}_{\omega, t} - \gamma_t \frac{\mu_t}{\Delta t} \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

$$\hat{\mathbf{q}}_{\text{est}, t} = \hat{\mathbf{q}}_{\text{est}, t-1} + \left((1 - \gamma_t) \hat{\mathbf{q}}_{\omega, t} - \gamma_t \alpha \left\| \hat{\mathbf{q}}_{\omega, t} \right\| \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

$$\{ \|\hat{\mathbf{q}}_{\omega, t}\| : 0 \sim 13 \}$$

0.01~0.08
0.01~0.08

Gyroscope bias drift compensation

$$s_{\omega_b,t} = \xi \sum_t s_{\omega_e,t} \Delta t$$
$$\hat{\hat{s}}\dot{q}_{\omega,t} = \frac{1}{2} \hat{\hat{s}}\dot{q}_{est,t-1} \otimes s_{\omega_t}$$
$$2 \hat{\hat{s}}\dot{q}_{\omega,t} = \hat{\hat{s}}\dot{q}_{est,t-1} \otimes s_{\omega_t}$$
$$2 \hat{\hat{s}}\dot{q}_{est,t-1}^* \otimes \hat{\hat{s}}\dot{q}_{\omega,t} = \hat{\hat{s}}\dot{q}_{est,t-1}^* \otimes \hat{\hat{s}}\dot{q}_{est,t-1} \otimes s_{\omega_t}$$
$$2 \hat{\hat{s}}\dot{q}_{est,t-1}^* \otimes \hat{\hat{s}}\dot{q}_{\omega,t} = s_{\omega_t}$$
$$s_{\omega_e,t} = 2 \hat{\hat{s}}\dot{q}_{est,t-1}^* \otimes \hat{\hat{s}}\dot{q}_{e,t}$$
$$\begin{pmatrix} s_{\omega_e x} \\ s_{\omega_e y} \\ s_{\omega_e z} \end{pmatrix} = 2 \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} \otimes \begin{pmatrix} \hat{\hat{s}}\dot{q}_1 & \hat{\hat{s}}\dot{q}_2 & \hat{\hat{s}}\dot{q}_3 & \hat{\hat{s}}\dot{q}_4 \end{pmatrix} = 2 \begin{pmatrix} q_1 * \hat{\hat{s}}\dot{q}_2 - q_2 * \hat{\hat{s}}\dot{q}_1 - q_3 * \hat{\hat{s}}\dot{q}_4 + q_4 * \hat{\hat{s}}\dot{q}_3 \\ q_1 * \hat{\hat{s}}\dot{q}_3 + q_2 * \hat{\hat{s}}\dot{q}_4 - q_3 * \hat{\hat{s}}\dot{q}_1 - q_4 * \hat{\hat{s}}\dot{q}_2 \\ q_1 * \hat{\hat{s}}\dot{q}_4 - q_2 * \hat{\hat{s}}\dot{q}_3 + q_3 * \hat{\hat{s}}\dot{q}_2 - q_4 * \hat{\hat{s}}\dot{q}_1 \end{pmatrix}$$
$$s_{\omega_c,t} = s_{\omega_t} - s_{\omega_b,t}$$
$$\begin{pmatrix} w_{b x} \\ w_{b y} \\ w_{b z} \end{pmatrix} = \begin{pmatrix} w_{b x} \\ w_{b y} \\ w_{b z} \end{pmatrix} + \begin{pmatrix} w_{e x} \\ w_{e y} \\ w_{e z} \end{pmatrix} * \Delta t * \xi$$
$$\begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} - \begin{pmatrix} w_{b x} \\ w_{b y} \\ w_{b z} \end{pmatrix}$$

Filter gains

$$\beta = \left| \frac{1}{2} \dot{q} \otimes \begin{pmatrix} 0 & \tilde{\omega}_\beta & \tilde{\omega}_\beta & \tilde{\omega}_\beta \end{pmatrix} \right| = \sqrt{\frac{3}{4}} \tilde{\omega}_\beta$$
$$\xi = \xi \sqrt{\frac{3}{4}} \tilde{\omega}_\xi$$

F

$$\begin{pmatrix} xx & yx & zx \\ xy & yy & zy \\ xz & yz & zz \end{pmatrix} = \begin{pmatrix} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 + q_1 q_4) & 2 (q_2 q_4 - q_1 q_3) \\ 2 (q_2 q_3 - q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 + q_1 q_2) \\ 2 (q_2 q_4 + q_1 q_3) & 2 (q_3 q_4 - q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{pmatrix}^T;$$

xx + yx + zx // FullSimplify

$$-1 + 2 q_1^2 + 2 q_1 (q_3 - q_4) + 2 q_2 (q_2 + q_3 + q_4)$$
$$-1 + 2 (q_1^2 + q_1 (q_3 - q_4) + q_2 (q_2 + q_3 + q_4))$$

xy + yy + zy // FullSimplify

$$-1 + 2 (q_1^2 + q_1 (-q_2 + q_4) + q_3 (q_2 + q_3 + q_4))$$

xz + yz + zz // FullSimplify

$$-1 + 2 (q_1^2 + q_1 (q_2 - q_3) + q_4 (q_2 + q_3 + q_4))$$