

An efficient orientation filter for inertial and inertial/magnetic sensor arrays

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Magnetic sensor Soft/Hard Iron

Quaternion representation

Filter derivation

Orientation

Filter Readjustment

Orientation from vector observations

$${}^{\mathbb{S}}\hat{\mathbf{q}} = (q_1 \quad q_2 \quad q_3 \quad q_4)$$

$${}^{\mathbb{E}}\hat{\mathbf{d}} = (0 \quad d_x \quad d_y \quad d_z)$$

$${}^{\mathbb{S}}\hat{\mathbf{s}} = (0 \quad s_x \quad s_y \quad s_z)$$

$$\mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}] = {}^{\mathbb{S}}\hat{\mathbf{q}}^* \otimes {}^{\mathbb{E}}\hat{\mathbf{d}} \otimes {}^{\mathbb{S}}\hat{\mathbf{q}} - {}^{\mathbb{S}}\hat{\mathbf{s}}$$

$$\min_{{}^{\mathbb{S}}\hat{\mathbf{q}} \in \mathbb{R}^4} \mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}]$$

$${}^{\mathbb{S}}\hat{\mathbf{q}}_{k+1} = {}^{\mathbb{S}}\hat{\mathbf{q}}_k - \mu \frac{\nabla \mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}]}{|\nabla \mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}|}, \quad k = 0, 1, \dots, n$$

$$\nabla \mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}_k, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}] = \mathbf{J}^{\mathbf{T}}[{}^{\mathbb{S}}\hat{\mathbf{q}}_k, {}^{\mathbb{E}}\hat{\mathbf{d}}] \mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}_k, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}]$$

■ A objective function

$${}^{\mathbb{S}}\hat{\mathbf{q}}^* \otimes {}^{\mathbb{E}}\hat{\mathbf{d}} \otimes {}^{\mathbb{S}}\hat{\mathbf{q}} = (q_1 \quad -q_2 \quad -q_3 \quad -q_4) \otimes (0 \quad d_x \quad d_y \quad d_z) \otimes (q_1 \quad q_2 \quad q_3 \quad q_4)$$

$$= (q_1 \quad -q_2 \quad -q_3 \quad -q_4) \cdot \begin{pmatrix} 0 & d_x & d_y & d_z \\ -d_x & 0 & -d_z & d_y \\ -d_y & d_z & 0 & -d_x \\ -d_z & -d_y & d_x & 0 \end{pmatrix} \cdot \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \\ -q_2 & q_1 & -q_4 & q_3 \\ -q_3 & q_4 & q_1 & -q_2 \\ -q_4 & -q_3 & q_2 & q_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2(d_x(\frac{1}{2} - q_3^2 - q_4^2) + d_y(q_2 q_3 + q_1 q_4) + d_z(q_2 q_4 - q_1 q_3)) \\ 2(d_x(q_2 q_3 - q_1 q_4) + d_y(\frac{1}{2} - q_2^2 - q_4^2) + d_z(q_3 q_4 + q_1 q_2)) \\ 2(d_x(q_2 q_4 + q_1 q_3) + d_y(q_3 q_4 - q_1 q_2) + d_z(\frac{1}{2} - q_2^2 - q_3^2)) \end{pmatrix}^{\mathbf{T}}$$

$$\mathbf{f}[{}^{\mathbb{S}}\hat{\mathbf{q}}_k, {}^{\mathbb{E}}\hat{\mathbf{d}}, {}^{\mathbb{S}}\hat{\mathbf{s}}] = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 2 d_x (\frac{1}{2} - q_3^2 - q_4^2) + 2 d_y (q_2 q_3 + q_1 q_4) + 2 d_z (q_2 q_4 - q_1 q_3) - s_x \\ 2 d_x (q_2 q_3 - q_1 q_4) + 2 d_y (\frac{1}{2} - q_2^2 - q_4^2) + 2 d_z (q_3 q_4 + q_1 q_2) - s_y \\ 2 d_x (q_2 q_4 + q_1 q_3) + 2 d_y (q_3 q_4 - q_1 q_2) + 2 d_z (\frac{1}{2} - q_2^2 - q_3^2) - s_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}[{}^{\mathbb{S}}\hat{\mathbf{q}}_k, {}^{\mathbb{E}}\hat{\mathbf{d}}] = \begin{pmatrix} \partial_{q_1} f_1 & \partial_{q_2} f_1 & \partial_{q_3} f_1 & \partial_{q_4} f_1 \\ \partial_{q_1} f_2 & \partial_{q_2} f_2 & \partial_{q_3} f_2 & \partial_{q_4} f_2 \\ \partial_{q_1} f_3 & \partial_{q_2} f_3 & \partial_{q_3} f_3 & \partial_{q_4} f_3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 d_y q_4 - 2 d_z q_3 & 2 d_y q_3 + 2 d_z q_4 & -4 d_x q_3 + 2 d_y q_2 - 2 d_z q_1 & -4 d_x q_4 + 2 d_y q_1 + 2 d_z q_2 \\ -2 d_x q_4 + 2 d_z q_2 & 2 d_x q_3 - 4 d_y q_2 + 2 d_z q_1 & 2 d_x q_2 + 2 d_z q_4 & -2 d_x q_1 - 4 d_y q_4 + 2 d_z q_3 \\ 2 d_x q_3 - 2 d_y q_2 & 2 d_x q_4 - 2 d_y q_1 - 4 d_z q_2 & 2 d_x q_1 + 2 d_y q_4 - 4 d_z q_3 & 2 d_x q_2 + 2 d_y q_3 \end{pmatrix}$$

The direction of gravity (the vertical axis, z axis)

$${}^{\mathbb{E}}\hat{\mathbf{g}} = (0 \quad 0 \quad 0 \quad g_z)$$

$${}^{\mathbb{S}}\hat{\mathbf{a}} = (0 \quad a_x \quad a_y \quad a_z)$$

$$\mathbf{f}_g[{}^{\mathbb{S}}\hat{\mathbf{q}}, {}^{\mathbb{S}}\hat{\mathbf{a}}] = \begin{pmatrix} 2 g_z (q_2 q_4 - q_1 q_3) - a_x \\ 2 g_z (q_1 q_2 + q_3 q_4) - a_y \\ 2 g_z (\frac{1}{2} - q_2^2 - q_3^2) - a_z \end{pmatrix}$$

$$\mathbf{J}_g[{}^{\mathbb{S}}\hat{\mathbf{q}}] = \begin{pmatrix} -2 q_3 & 2 q_4 & -2 q_1 & 2 q_2 \\ 2 q_2 & 2 q_1 & 2 q_4 & 2 q_3 \\ 0 & -4 q_2 & -4 q_3 & 0 \end{pmatrix}$$

The earth's magnetic field (one horizontal axis, the vertical axis)

$${}^{\mathbb{E}}\hat{\mathbf{b}} = (0 \quad b_x \quad 0 \quad b_z)$$

$${}^{\mathbb{S}}\hat{\mathbf{m}} = (0 \quad m_x \quad m_y \quad m_z)$$

$$\mathbf{f}_b[{}^{\mathbb{S}}\hat{\mathbf{q}}, {}^{\mathbb{E}}\hat{\mathbf{b}}, {}^{\mathbb{S}}\hat{\mathbf{m}}] = \begin{pmatrix} 2 b_x (\frac{1}{2} - q_3^2 - q_4^2) + 2 b_z (q_2 q_4 - q_1 q_3) - m_x \\ 2 b_x (q_2 q_3 - q_1 q_4) + 2 b_z (q_1 q_2 + q_3 q_4) - m_y \\ 2 b_x (q_1 q_3 + q_2 q_4) + 2 b_z (\frac{1}{2} - q_2^2 - q_3^2) - m_z \end{pmatrix}$$

$$\mathbf{J}_b[{}^{\mathbb{S}}\hat{\mathbf{q}}, {}^{\mathbb{E}}\hat{\mathbf{b}}] = \begin{pmatrix} -2 b_z q_3 & 2 b_z q_4 & -4 b_x q_3 - 2 b_z q_1 & -4 b_x q_4 + 2 b_z q_2 \\ -2 b_x q_4 + 2 b_z q_2 & 2 b_x q_3 + 2 b_z q_1 & 2 b_x q_2 + 2 b_z q_4 & -2 b_x q_1 + 2 b_z q_3 \\ 2 b_x q_3 & 2 b_x q_4 - 4 b_z q_2 & 2 b_x q_1 - 4 b_z q_3 & 2 b_x q_2 \end{pmatrix}$$

The gradient of the solution surface (General form)

$$\nabla \mathbf{f}[\hat{\mathbf{e}}\hat{\mathbf{q}}_k, {}^E\hat{\mathbf{d}}, {}^S\hat{\mathbf{s}}] = \mathbf{J}[\hat{\mathbf{e}}\hat{\mathbf{q}}_k, {}^E\hat{\mathbf{d}}]^T \cdot \mathbf{f}[\hat{\mathbf{e}}\hat{\mathbf{q}}_k, {}^E\hat{\mathbf{d}}, {}^S\hat{\mathbf{s}}]$$

■ The objective function

$$\mathbf{f}_{g,b}[\hat{\mathbf{e}}\hat{\mathbf{q}}, {}^S\hat{\mathbf{a}}, {}^E\hat{\mathbf{b}}, {}^S\hat{\mathbf{m}}] = \begin{pmatrix} \mathbf{f}_g[\hat{\mathbf{e}}\hat{\mathbf{q}}, {}^S\hat{\mathbf{a}}] \\ \mathbf{f}_b[\hat{\mathbf{e}}\hat{\mathbf{q}}, {}^E\hat{\mathbf{b}}, {}^S\hat{\mathbf{m}}] \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{pmatrix} = \begin{pmatrix} 2(q_2 q_4 - q_1 q_3) - a_x \\ 2(q_1 q_2 + q_3 q_4) - a_y \\ 2(\frac{1}{2} - q_2^2 - q_3^2) - a_z \\ 2b_x(\frac{1}{2} - q_3^2 - q_4^2) + 2b_z(q_2 q_4 - q_1 q_3) - m_x \\ 2b_x(q_2 q_3 - q_1 q_4) + 2b_z(q_1 q_2 + q_3 q_4) - m_y \\ 2b_x(q_1 q_3 + q_2 q_4) + 2b_z(\frac{1}{2} - q_2^2 - q_3^2) - m_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}_{g,b}[\hat{\mathbf{e}}\hat{\mathbf{q}}, {}^E\hat{\mathbf{b}}] = \begin{pmatrix} \mathbf{J}_g^T[\hat{\mathbf{e}}\hat{\mathbf{q}}] \\ \mathbf{J}_b^T[\hat{\mathbf{e}}\hat{\mathbf{q}}, {}^E\hat{\mathbf{b}}] \end{pmatrix}$$

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \mathbf{J}_{13} & \mathbf{J}_{14} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \mathbf{J}_{23} & \mathbf{J}_{24} \\ \mathbf{J}_{31} & \mathbf{J}_{32} & \mathbf{J}_{33} & \mathbf{J}_{34} \\ \mathbf{J}_{41} & \mathbf{J}_{42} & \mathbf{J}_{43} & \mathbf{J}_{44} \\ \mathbf{J}_{51} & \mathbf{J}_{52} & \mathbf{J}_{53} & \mathbf{J}_{54} \\ \mathbf{J}_{61} & \mathbf{J}_{62} & \mathbf{J}_{63} & \mathbf{J}_{64} \end{pmatrix} = \begin{pmatrix} -2q_3 & 2q_4 & -2q_1 & 2q_2 \\ 2q_2 & 2q_1 & 2q_4 & 2q_3 \\ 0 & -4q_2 & -4q_3 & 0 \\ -2b_x q_3 & 2b_z q_4 & -4b_x q_3 - 2b_z q_1 & -4b_x q_4 + 2b_z q_2 \\ -2b_x q_4 + 2b_z q_2 & 2b_x q_3 + 2b_z q_1 & 2b_x q_2 + 2b_z q_4 & -2b_x q_1 + 2b_z q_3 \\ 2b_x q_3 & 2b_x q_4 - 4b_z q_2 & 2b_x q_1 - 4b_z q_3 & 2b_x q_2 \end{pmatrix}$$

■ gradient

$$\nabla \mathbf{f}_{g,b} = \mathbf{J}_{g,b}[\hat{\mathbf{e}}\hat{\mathbf{q}}, {}^E\hat{\mathbf{b}}]^T \cdot \mathbf{f}_{g,b}[\hat{\mathbf{e}}\hat{\mathbf{q}}, {}^S\hat{\mathbf{a}}, {}^E\hat{\mathbf{b}}, {}^S\hat{\mathbf{m}}]$$

$$\begin{pmatrix} \nabla \mathbf{f}_1 \\ \nabla \mathbf{f}_2 \\ \nabla \mathbf{f}_3 \\ \nabla \mathbf{f}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \mathbf{J}_{13} & \mathbf{J}_{14} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \mathbf{J}_{23} & \mathbf{J}_{24} \\ \mathbf{J}_{31} & \mathbf{J}_{32} & \mathbf{J}_{33} & \mathbf{J}_{34} \\ \mathbf{J}_{41} & \mathbf{J}_{42} & \mathbf{J}_{43} & \mathbf{J}_{44} \\ \mathbf{J}_{51} & \mathbf{J}_{52} & \mathbf{J}_{53} & \mathbf{J}_{54} \\ \mathbf{J}_{61} & \mathbf{J}_{62} & \mathbf{J}_{63} & \mathbf{J}_{64} \end{pmatrix}^T \cdot \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \mathbf{J}_{11} + \mathbf{f}_2 \mathbf{J}_{21} + \mathbf{f}_3 \mathbf{J}_{31} + \mathbf{f}_4 \mathbf{J}_{41} + \mathbf{f}_5 \mathbf{J}_{51} + \mathbf{f}_6 \mathbf{J}_{61} \\ \mathbf{f}_1 \mathbf{J}_{12} + \mathbf{f}_2 \mathbf{J}_{22} + \mathbf{f}_3 \mathbf{J}_{32} + \mathbf{f}_4 \mathbf{J}_{42} + \mathbf{f}_5 \mathbf{J}_{52} + \mathbf{f}_6 \mathbf{J}_{62} \\ \mathbf{f}_1 \mathbf{J}_{13} + \mathbf{f}_2 \mathbf{J}_{23} + \mathbf{f}_3 \mathbf{J}_{33} + \mathbf{f}_4 \mathbf{J}_{43} + \mathbf{f}_5 \mathbf{J}_{53} + \mathbf{f}_6 \mathbf{J}_{63} \\ \mathbf{f}_1 \mathbf{J}_{14} + \mathbf{f}_2 \mathbf{J}_{24} + \mathbf{f}_3 \mathbf{J}_{34} + \mathbf{f}_4 \mathbf{J}_{44} + \mathbf{f}_5 \mathbf{J}_{54} + \mathbf{f}_6 \mathbf{J}_{64} \end{pmatrix}$$

$$\begin{pmatrix} \hat{\mathbf{e}}\dot{\hat{\mathbf{q}}}_1 \\ \hat{\mathbf{e}}\dot{\hat{\mathbf{q}}}_2 \\ \hat{\mathbf{e}}\dot{\hat{\mathbf{q}}}_3 \\ \hat{\mathbf{e}}\dot{\hat{\mathbf{q}}}_4 \end{pmatrix} = \begin{pmatrix} \frac{\nabla \mathbf{f}_1}{|\nabla \mathbf{f}|} \\ \frac{\nabla \mathbf{f}_2}{|\nabla \mathbf{f}|} \\ \frac{\nabla \mathbf{f}_3}{|\nabla \mathbf{f}|} \\ \frac{\nabla \mathbf{f}_4}{|\nabla \mathbf{f}|} \end{pmatrix}$$

The estimate orientation $\hat{\mathbf{e}}\hat{\mathbf{q}}_{\nabla, t}$

$$\hat{\mathbf{e}}\hat{\mathbf{q}}_{\nabla, t} = \hat{\mathbf{e}}\hat{\mathbf{q}}_{\text{est}, t-1} - \mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|}$$

$$\mu_t = \alpha \mid \hat{\mathbf{e}}\hat{\mathbf{q}}_{\omega, t} \mid \Delta t, \alpha > 1$$

Orientation from angular rate

$${}^S\omega = (\mathbf{0} \quad \omega_x \quad \omega_y \quad \omega_z)$$

$$\hat{\mathbf{e}}\hat{\mathbf{q}}_{\omega, t} = \frac{1}{2} \hat{\mathbf{e}}\hat{\mathbf{q}}_{\text{est}, t-1} \otimes {}^S\omega_t$$

$$(\mathbf{q}_{\omega 1} \quad \mathbf{q}_{\omega 2} \quad \mathbf{q}_{\omega 3} \quad \mathbf{q}_{\omega 4}) = \frac{1}{2} (\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3 \quad \mathbf{q}_4) \otimes (\mathbf{0} \quad \omega_x \quad \omega_y \quad \omega_z)$$

$$\hat{\mathbf{e}}\hat{\mathbf{q}}_{\omega, t} = \hat{\mathbf{e}}\hat{\mathbf{q}}_{\text{est}, t-1} + \hat{\mathbf{e}}\hat{\mathbf{q}}_{\omega, t} \Delta t$$

Magnetic distortion compensation

$${}^E\hat{\mathbf{h}}_t = (\mathbf{0} \quad h_x \quad h_y \quad h_z) = \hat{\mathbf{e}}\hat{\mathbf{q}}_{\text{est}, t-1} \otimes {}^S\hat{\mathbf{m}}_t \otimes \hat{\mathbf{e}}\hat{\mathbf{q}}_{\text{est}, t-1}^*$$

$$\begin{aligned}
&= \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{0} & m_x & m_y & m_z \end{pmatrix} \otimes \begin{pmatrix} \mathbf{q}_1 & -\mathbf{q}_2 & -\mathbf{q}_3 & -\mathbf{q}_4 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{0} & m_x & m_y & m_z \\ -m_x & \mathbf{0} & -m_z & m_y \\ -m_y & m_z & \mathbf{0} & -m_x \\ -m_z & -m_y & m_x & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{q}_1 & -\mathbf{q}_2 & -\mathbf{q}_3 & -\mathbf{q}_4 \\ \mathbf{q}_2 & \mathbf{q}_1 & \mathbf{q}_4 & -\mathbf{q}_3 \\ \mathbf{q}_3 & -\mathbf{q}_4 & \mathbf{q}_1 & \mathbf{q}_2 \\ \mathbf{q}_4 & \mathbf{q}_3 & -\mathbf{q}_2 & \mathbf{q}_1 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0} \\ 2 \left(m_x \left(\frac{1}{2} - \mathbf{q}_3^2 - \mathbf{q}_4^2 \right) + m_y \left(\mathbf{q}_2 \mathbf{q}_3 - \mathbf{q}_1 \mathbf{q}_4 \right) + m_z \left(\mathbf{q}_1 \mathbf{q}_3 + \mathbf{q}_2 \mathbf{q}_4 \right) \right) \\ 2 \left(m_x \left(\mathbf{q}_2 \mathbf{q}_3 + \mathbf{q}_1 \mathbf{q}_4 \right) + m_y \left(\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_4^2 \right) + m_z \left(-\mathbf{q}_1 \mathbf{q}_2 + \mathbf{q}_3 \mathbf{q}_4 \right) \right) \\ 2 \left(m_x \left(-\mathbf{q}_1 \mathbf{q}_3 + \mathbf{q}_2 \mathbf{q}_4 \right) + m_y \left(\mathbf{q}_1 \mathbf{q}_2 + \mathbf{q}_3 \mathbf{q}_4 \right) + m_z \left(\frac{1}{2} - \mathbf{q}_2^2 - \mathbf{q}_3^2 \right) \right) \end{pmatrix}^T \\
{}^E \hat{\mathbf{b}}_t &= \begin{pmatrix} \mathbf{0} & \sqrt{h_x^2 + h_y^2} & \mathbf{0} & h_z \end{pmatrix} \\
\begin{pmatrix} \mathbf{0} & b_x & b_y & b_z \end{pmatrix} &= \begin{pmatrix} \mathbf{0} & \sqrt{h_x^2 + h_y^2} & \mathbf{0} & h_z \end{pmatrix}
\end{aligned}$$

Filter fusion algorithm

$$\blacksquare \quad (1 - \gamma_t) \beta = \gamma_t \frac{\mu_t}{\Delta t}$$

$$\blacksquare \quad {}^E \hat{\mathbf{q}}_{\omega, t} = {}^E \hat{\mathbf{q}}_{\text{est}, t-1} + {}^E \dot{\mathbf{q}}_{\omega, t} \Delta t$$

$${}^E \dot{\mathbf{q}}_{\omega, t} = \frac{1}{2} {}^E \hat{\mathbf{q}}_{\text{est}, t-1} \otimes {}^S \omega_t$$

$$\blacksquare \quad {}^E \hat{\mathbf{q}}_{\nabla, t} = {}^E \hat{\mathbf{q}}_{\text{est}, t-1} - \mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|}$$

$$\mu_t = \alpha |\dot{{}^E \hat{\mathbf{q}}_{\omega, t}}| \Delta t, \quad \alpha > 1$$

$$\blacksquare \quad {}^E \hat{\mathbf{q}}_{\text{est}, t} = \gamma_t {}^E \hat{\mathbf{q}}_{\nabla, t} + (1 - \gamma_t) {}^E \hat{\mathbf{q}}_{\omega, t}, \quad 0 \leq \gamma_t \leq 1$$

$${}^E \hat{\mathbf{q}}_{\text{est}, t} = \gamma_t \left({}^E \hat{\mathbf{q}}_{\text{est}, t-1} - \mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) + (1 - \gamma_t) ({}^E \hat{\mathbf{q}}_{\text{est}, t-1} + {}^E \dot{\mathbf{q}}_{\omega, t} \Delta t)$$

$${}^E \hat{\mathbf{q}}_{\text{est}, t} = {}^E \hat{\mathbf{q}}_{\text{est}, t-1} - \gamma_t \mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} + (1 - \gamma_t) {}^E \dot{\mathbf{q}}_{\omega, t} \Delta t$$

$${}^E \hat{\mathbf{q}}_{\text{est}, t} = {}^E \hat{\mathbf{q}}_{\text{est}, t-1} + \left((1 - \gamma_t) {}^E \dot{\mathbf{q}}_{\omega, t} - \gamma_t \frac{\mu_t}{\Delta t} \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

$${}^E \hat{\mathbf{q}}_{\text{est}, t} = {}^E \hat{\mathbf{q}}_{\text{est}, t-1} + \left((1 - \gamma_t) {}^E \dot{\mathbf{q}}_{\omega, t} - \gamma_t \alpha \left| {}^E \dot{\mathbf{q}}_{\omega, t} \right| \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

Gyroscope bias drift compensation

$${}^S \omega_{b, t} = \zeta \sum_t {}^S \omega_{e, t} \Delta t$$

$${}^E \dot{\mathbf{q}}_{\omega, t} = \frac{1}{2} {}^E \hat{\mathbf{q}}_{\text{est}, t-1} \otimes {}^S \omega_t$$

$$2 {}^E \dot{\mathbf{q}}_{\omega, t} = {}^E \hat{\mathbf{q}}_{\text{est}, t-1} \otimes {}^S \omega_t$$

$$2 {}^E \hat{\mathbf{q}}_{\text{est}, t-1}^* \otimes {}^E \dot{\mathbf{q}}_{\omega, t} = {}^E \hat{\mathbf{q}}_{\text{est}, t-1}^* \otimes {}^E \hat{\mathbf{q}}_{\text{est}, t-1} \otimes {}^S \omega_t$$

$$2 {}^E \hat{\mathbf{q}}_{\text{est}, t-1}^* \otimes {}^E \dot{\mathbf{q}}_{\omega, t} = {}^S \omega_t$$

$${}^S \omega_{e, t} = 2 {}^E \hat{\mathbf{q}}_{\text{est}, t-1}^* \otimes {}^E \dot{\mathbf{q}}_{e, t}$$

$$\begin{pmatrix} {}^S \omega_{e, x} \\ {}^S \omega_{e, y} \\ {}^S \omega_{e, z} \end{pmatrix} = 2 \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \mathbf{q}_4 \end{pmatrix} \otimes \begin{pmatrix} {}^E \dot{\mathbf{q}}_1 & {}^E \dot{\mathbf{q}}_2 & {}^E \dot{\mathbf{q}}_3 & {}^E \dot{\mathbf{q}}_4 \end{pmatrix} = 2 \begin{pmatrix} \mathbf{q}_1 * {}^E \dot{\mathbf{q}}_2 - \mathbf{q}_2 * {}^E \dot{\mathbf{q}}_1 - \mathbf{q}_3 * {}^E \dot{\mathbf{q}}_4 + \mathbf{q}_4 * {}^E \dot{\mathbf{q}}_3 \\ \mathbf{q}_1 * {}^E \dot{\mathbf{q}}_3 + \mathbf{q}_2 * {}^E \dot{\mathbf{q}}_4 - \mathbf{q}_3 * {}^E \dot{\mathbf{q}}_1 - \mathbf{q}_4 * {}^E \dot{\mathbf{q}}_2 \\ \mathbf{q}_1 * {}^E \dot{\mathbf{q}}_4 - \mathbf{q}_2 * {}^E \dot{\mathbf{q}}_3 + \mathbf{q}_3 * {}^E \dot{\mathbf{q}}_2 - \mathbf{q}_4 * {}^E \dot{\mathbf{q}}_1 \end{pmatrix}$$

$${}^S \omega_{c, t} = {}^S \omega_t - {}^S \omega_{b, t}$$

$$\begin{pmatrix} w_{b, x} \\ w_{b, y} \\ w_{b, z} \end{pmatrix} = \begin{pmatrix} w_{b, x} \\ w_{b, y} \\ w_{b, z} \end{pmatrix} + \begin{pmatrix} w_{e, x} \\ w_{e, y} \\ w_{e, z} \end{pmatrix} * \Delta t * \zeta$$

$$\begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} - \begin{pmatrix} w_{b, x} \\ w_{b, y} \\ w_{b, z} \end{pmatrix}$$

Filter gains

$$\beta = \left| \frac{1}{2} \mathbf{q} \otimes \left(\mathbf{0} \quad \tilde{\omega}_\beta \quad \tilde{\omega}_\beta \quad \tilde{\omega}_\beta \right) \right| = \sqrt{\frac{3}{4}} \tilde{\omega}_\beta$$

$$\xi = \xi \sqrt{\frac{3}{4}} \tilde{\omega}_\xi$$