

Direction Cosine

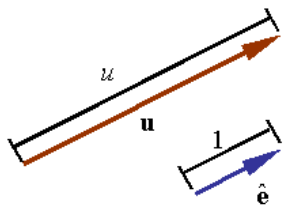
단위벡터의 내적(dot product) → 두 벡터 사이 각의 코사인 값

- Inner Product Space

$$\vec{x} \cdot \vec{y} = |\vec{x}| |\vec{y}| \cos[\theta]$$

$$\cos[\theta] = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}| |\vec{y}|}$$

- Unit Vector



$$|\hat{e}| = 1$$

$$\hat{e} = \frac{\vec{u}}{|\vec{u}|} = \frac{\vec{u}}{u}$$

$$\vec{u} = |\vec{u}| \hat{e} = u \hat{e}$$

$$|\hat{e}_x| = 1$$

$$|\hat{e}_y| = 1$$

$$|\hat{e}_z| = 1$$

$$\hat{e}_x \cdot \hat{e}_x = 1$$

$$\hat{e}_y \cdot \hat{e}_y = 1$$

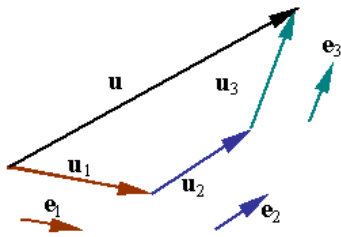
$$\hat{e}_z \cdot \hat{e}_z = 1$$

$$\hat{e}_x \cdot \hat{e}_y = \hat{e}_y \cdot \hat{e}_x = 0$$

$$\hat{e}_y \cdot \hat{e}_z = \hat{e}_z \cdot \hat{e}_y = 0$$

$$\hat{e}_z \cdot \hat{e}_x = \hat{e}_x \cdot \hat{e}_z = 0$$

- Base vectors and vector components

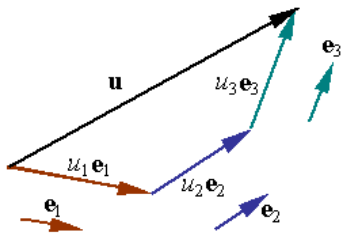


$$\vec{u} = \vec{u}_1 + \vec{u}_2 + \vec{u}_3$$

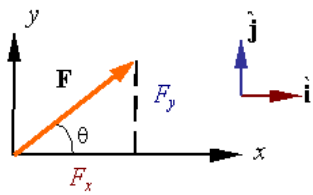
$$\vec{u}_1 = u_1 \vec{e}_1$$

$$\vec{u}_2 = u_2 \vec{e}_2$$

$$\vec{u}_3 = u_3 \vec{e}_3$$



$$\vec{u} = u_1 \vec{e}_1 + u_2 \vec{e}_2 + u_3 \vec{e}_3$$



$$\vec{F} = F_x \hat{i} + F_y \hat{j}$$

$$F = \sqrt{F_x^2 + F_y^2}$$

$$F_x = F \cos[\theta]$$

$$F_y = F \sin[\theta]$$

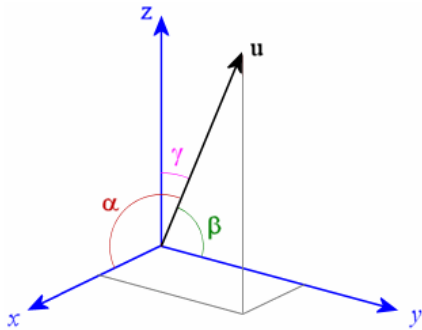
- Vector to Matrix

In[18]:=

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{px, py, pz} // MatrixForm
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$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

- Representation of orientation



$$a^2 + b^2 + c^2 = \cos^2[\alpha] + \cos^2[\beta] + \cos^2[\gamma] = 1$$

$$\vec{v} = \{v_x, v_y, v_z\} \cdot \{\vec{e}_x, \vec{e}_y, \vec{e}_z\} = v_x \vec{e}_x + v_y \vec{e}_y + v_z \vec{e}_z$$

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$a = \cos[\alpha] = \frac{\vec{v} \cdot \vec{e}_x}{|\vec{v}|} = \frac{(v_x \vec{e}_x + v_y \vec{e}_y + v_z \vec{e}_z) \cdot \vec{e}_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_x}{|\vec{v}|}$$

$$b = \cos[\beta] = \frac{\vec{v} \cdot \vec{e}_y}{|\vec{v}|} = \frac{(v_x \vec{e}_x + v_y \vec{e}_y + v_z \vec{e}_z) \cdot \vec{e}_y}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_y}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_y}{|\vec{v}|}$$

$$c = \cos[\gamma] = \frac{\vec{v} \cdot \vec{e}_z}{|\vec{v}|} = \frac{(v_x \vec{e}_x + v_y \vec{e}_y + v_z \vec{e}_z) \cdot \vec{e}_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}} = \frac{v_z}{|\vec{v}|}$$

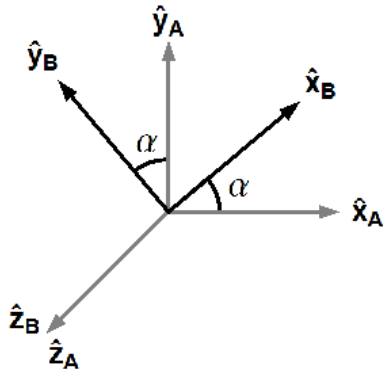
$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{v_x}{|\vec{v}|} \vec{e}_x + \frac{v_y}{|\vec{v}|} \vec{e}_y + \frac{v_z}{|\vec{v}|} \vec{e}_z$$

$$= \cos[\alpha] \vec{e}_x + \cos[\beta] \vec{e}_y + \cos[\gamma] \vec{e}_z$$

$$= a \vec{e}_x + b \vec{e}_y + c \vec{e}_z$$

$$\vec{v} = |\vec{v}| \vec{u} = |\vec{v}| \cos[\alpha] \vec{e}_x + |\vec{v}| \cos[\beta] \vec{e}_y + |\vec{v}| \cos[\gamma] \vec{e}_z = v_x \vec{e}_x + v_y \vec{e}_y + v_z \vec{e}_z$$

- 방위의 표현: 회전행렬



좌표계 {A} 가 $\hat{z}_A$ 축을 기준으로 α 만큼 회전한 좌표계를 {B} 라고 할 경우 회전행렬 ${}^A_B R$

$${}^A_B R = \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix} = \begin{pmatrix} \cos[\alpha] & \cos[90^\circ + \alpha] & \cos[90^\circ] \\ \cos[90^\circ - \alpha] & \cos[\alpha] & \cos[90^\circ] \\ \cos[90^\circ] & \cos[90^\circ] & \cos[0^\circ] \end{pmatrix} = \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} = R_z[\alpha]$$

$$R_z[\psi] = \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y[\theta] = \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix}$$

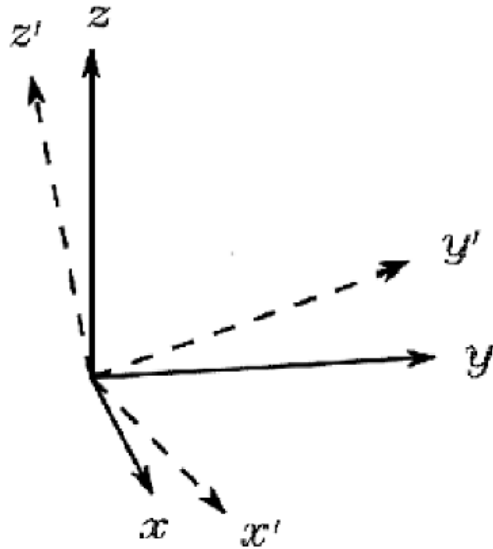
$$R_x[\phi] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix}$$

$${}^A_B R = \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix}$$

$${}^B_A R = \begin{pmatrix} \hat{x}_A \cdot \hat{x}_B & \hat{y}_A \cdot \hat{x}_B & \hat{z}_A \cdot \hat{x}_B \\ \hat{x}_A \cdot \hat{y}_B & \hat{y}_A \cdot \hat{y}_B & \hat{z}_A \cdot \hat{y}_B \\ \hat{x}_A \cdot \hat{z}_B & \hat{y}_A \cdot \hat{z}_B & \hat{z}_A \cdot \hat{z}_B \end{pmatrix}$$

$${}^A_B R^T = {}^B_A R$$

Representation of orientation



- Orientation

$$\mathbf{v} = \{v_x, v_y, v_z\}$$

$$\vec{v} = \mathbf{v} \cdot \{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\} = \{v_x, v_y, v_z\} \cdot \{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\} = v_x \hat{\mathbf{e}}_x + v_y \hat{\mathbf{e}}_y + v_z \hat{\mathbf{e}}_z$$

$$\hat{\mathbf{e}}_x = \frac{\hat{\mathbf{x}}}{|\hat{\mathbf{x}}|}$$

$$\hat{\mathbf{e}}_y = \frac{\hat{\mathbf{y}}}{|\hat{\mathbf{y}}|}$$

$$\hat{\mathbf{e}}_z = \frac{\hat{\mathbf{z}}}{|\hat{\mathbf{z}}|}$$

$$\mathbf{x} = \{x_x, x_y, x_z\}$$

$$\mathbf{y} = \{y_x, y_y, y_z\}$$

$$\mathbf{z} = \{z_x, z_y, z_z\}$$

$$\hat{\mathbf{x}} = \mathbf{x} \cdot \{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\} = x_x \hat{\mathbf{e}}_x + x_y \hat{\mathbf{e}}_y + x_z \hat{\mathbf{e}}_z = x_x \frac{\hat{\mathbf{x}}}{|\hat{\mathbf{x}}|} + x_y \frac{\hat{\mathbf{y}}}{|\hat{\mathbf{y}}|} + x_z \frac{\hat{\mathbf{z}}}{|\hat{\mathbf{z}}|}$$

$$\hat{\mathbf{y}} = \mathbf{y} \cdot \{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\} = y_x \hat{\mathbf{e}}_x + y_y \hat{\mathbf{e}}_y + y_z \hat{\mathbf{e}}_z$$

$$\vec{z} = z_{\kappa} \cdot \{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\} = z_{\kappa} \vec{e}_{\kappa} + z_Y \vec{e}_Y + z_s \vec{e}_s$$

$$|\vec{x}| = |\vec{x'}| = 1$$

$$|\vec{y}| = |\vec{y'}| = 1$$

$$|\vec{z}| = |\vec{z'}| = 1$$

$$\mathbf{x'} = \{x_{\kappa'}, x_Y', x_s'\}$$

$$\mathbf{y'} = \{y_{\kappa'}, y_Y', y_s'\}$$

$$\mathbf{z'} = \{z_{\kappa'}, z_Y', z_s'\}$$

$$\vec{x'} = \mathbf{x'} \cdot \{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\} = \{x_{\kappa'}, x_Y', x_s'\} \cdot \{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\} = \{x_{\kappa'}, x_Y', x_s'\} \cdot \left\{ \frac{\vec{x}}{|\vec{x}|}, \frac{\vec{y}}{|\vec{y}|}, \frac{\vec{z}}{|\vec{z}|} \right\}$$

$$= \{x_{\kappa'}, x_Y', x_s'\} \cdot \{\vec{x}, \vec{y}, \vec{z}\} = \vec{x} x_{\kappa'} + \vec{y} x_Y' + \vec{z} x_s'$$

$$= (x_{\kappa} \vec{e}_{\kappa} + x_Y \vec{e}_Y + x_s \vec{e}_s) x_{\kappa'} + (y_{\kappa} \vec{e}_{\kappa} + y_Y \vec{e}_Y + y_s \vec{e}_s) x_Y' + (z_{\kappa} \vec{e}_{\kappa} + z_Y \vec{e}_Y + z_s \vec{e}_s) x_s'$$

$$= x_{\kappa} x_{\kappa'} \vec{e}_{\kappa} + x_Y x_{\kappa'} \vec{e}_Y + x_s x_{\kappa'} \vec{e}_s + y_{\kappa} x_Y' \vec{e}_{\kappa} + y_Y x_Y' \vec{e}_Y + y_s x_Y' \vec{e}_s + z_{\kappa} x_s' \vec{e}_{\kappa} + z_Y x_s' \vec{e}_Y + z_s x_s' \vec{e}_s$$

$$= (x_{\kappa} x_{\kappa'} + y_{\kappa} x_Y' + z_{\kappa} x_s') \vec{e}_{\kappa} + (x_Y x_{\kappa'} + y_Y x_Y' + z_Y x_s') \vec{e}_Y + (x_s x_{\kappa'} + y_s x_Y' + z_s x_s') \vec{e}_s$$

$$= \{x_{\kappa} x_{\kappa'} + y_{\kappa} x_Y' + z_{\kappa} x_s', x_Y x_{\kappa'} + y_Y x_Y' + z_Y x_s', x_s x_{\kappa'} + y_s x_Y' + z_s x_s'\} \cdot \{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\}$$

$$\vec{y'} = \mathbf{y'} \cdot \{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\} = \{y_{\kappa'}, y_Y', y_s'\} \cdot \{\vec{x}, \vec{y}, \vec{z}\} = \vec{x} y_{\kappa'} + \vec{y} y_Y' + \vec{z} y_s'$$

$$= \{x_{\kappa} y_{\kappa'} + y_{\kappa} y_Y' + z_{\kappa} y_s', x_Y y_{\kappa'} + y_Y y_Y' + z_Y y_s', x_s y_{\kappa'} + y_s y_Y' + z_s y_s'\} \cdot \{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\}$$

$$\vec{z'} = \mathbf{z'} \cdot \{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\} = \{z_{\kappa'}, z_Y', z_s'\} \cdot \{\vec{x}, \vec{y}, \vec{z}\} = \vec{x} z_{\kappa'} + \vec{y} z_Y' + \vec{z} z_s'$$

$$= \{x_{\kappa} z_{\kappa'} + y_{\kappa} z_Y' + z_{\kappa} z_s', x_Y z_{\kappa'} + y_Y z_Y' + z_Y z_s', x_s z_{\kappa'} + y_s z_Y' + z_s z_s'\} \cdot \{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\}$$

■ ROTATION MATRIX

ln[22]:=

$$\{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

ln[23]:=

$$\vec{x} = \{x_{\kappa}, x_Y, x_s\} \cdot \{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\};$$

$$\vec{y} = \{y_{\kappa}, y_Y, y_s\} \cdot \{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\};$$

$$\vec{z} = \{z_{\kappa}, z_Y, z_s\} \cdot \{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\};$$

$$\vec{x'} = \{x_{\kappa'}, x_Y', x_s'\} \cdot \{\vec{e}_{\kappa}, \vec{e}_Y, \vec{e}_s\};$$

$$\overrightarrow{y} = \{y_x', y_y', y_z'\} \cdot \{\hat{e}_x, \hat{e}_y, \hat{e}_z\};$$

$$\overrightarrow{z} = \{z_x', z_y', z_z'\} \cdot \{\hat{e}_x, \hat{e}_y, \hat{e}_z\};$$

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$$B = \left\{ \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z} \right\} = \begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \\ \overrightarrow{z} \end{pmatrix} = \begin{pmatrix} x_x' & x_y' & x_z' \\ y_x' & y_y' & y_z' \\ z_x' & z_y' & z_z' \end{pmatrix};$$

$$A = \left\{ \hat{x}, \hat{y}, \hat{z} \right\} = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} x_x & x_y & x_z \\ y_x & y_y & y_z \\ z_x & z_y & z_z \end{pmatrix};$$

$$(B.(A^T))^T // MatrixForm$$

$$\begin{pmatrix} \overrightarrow{x}.\hat{x} & \overrightarrow{y}.\hat{x} & \overrightarrow{z}.\hat{x} \\ \overrightarrow{x}.\hat{y} & \overrightarrow{y}.\hat{y} & \overrightarrow{z}.\hat{y} \\ \overrightarrow{x}.\hat{z} & \overrightarrow{y}.\hat{z} & \overrightarrow{z}.\hat{z} \end{pmatrix} // MatrixForm$$

$${}_B R = \begin{pmatrix} \overrightarrow{x}.\hat{x} & \overrightarrow{y}.\hat{x} & \overrightarrow{z}.\hat{x} \\ \overrightarrow{x}.\hat{y} & \overrightarrow{y}.\hat{y} & \overrightarrow{z}.\hat{y} \\ \overrightarrow{x}.\hat{z} & \overrightarrow{y}.\hat{z} & \overrightarrow{z}.\hat{z} \end{pmatrix} = \left(\left\{ \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z} \right\} \cdot \left\{ \hat{x}, \hat{y}, \hat{z} \right\}^T \right)^T = \left(\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \\ \overrightarrow{z} \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}^T \right)^T = \left(\begin{pmatrix} x_x' & x_y' & x_z' \\ y_x' & y_y' & y_z' \\ z_x' & z_y' & z_z' \end{pmatrix} \cdot \begin{pmatrix} x_x & x_y & x_z \\ y_x & y_y & y_z \\ z_x & z_y & z_z \end{pmatrix}^T \right)^T$$

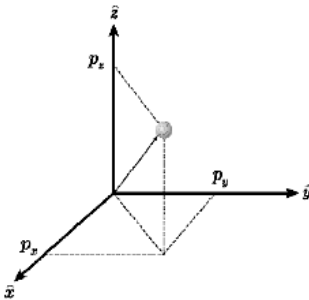
$$= \begin{pmatrix} x_x x_x' + x_y x_y' + x_z x_z' & x_x y_x' + x_y y_y' + x_z y_z' & x_x z_x' + x_y z_y' + x_z z_z' \\ y_x x_x' + y_y x_y' + y_z x_z' & y_x y_x' + y_y y_y' + y_z y_z' & y_x z_x' + y_y z_y' + y_z z_z' \\ z_x x_x' + z_y x_y' + z_z x_z' & z_x y_x' + z_y y_y' + z_z y_z' & z_x z_x' + z_y z_y' + z_z z_z' \end{pmatrix}$$

$$R.R^T = R^T.R = I$$

Analysis on Position And Orientation

- Cartesian Coordinate (직교좌표계)

unit vector



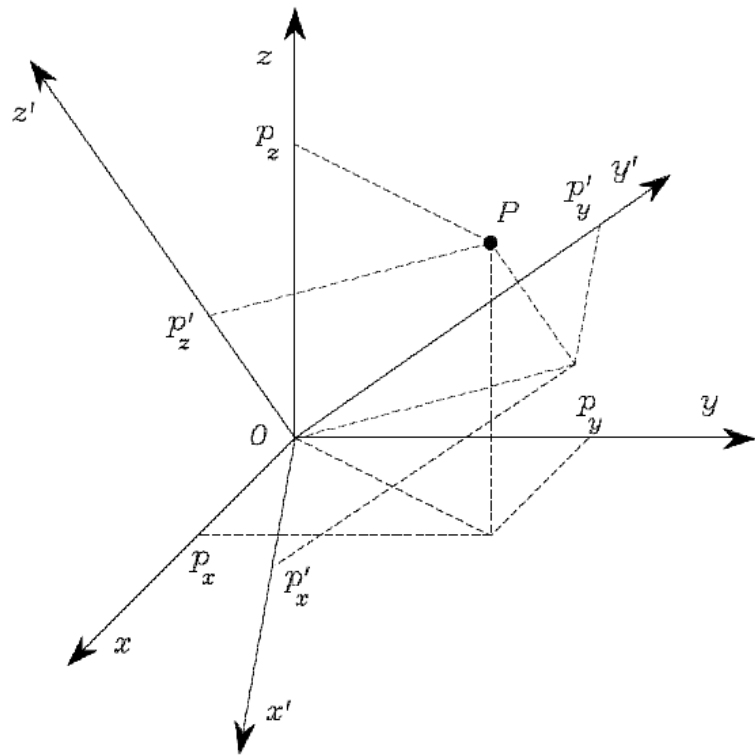
- One Point P Vector

$$P = \{p_x, p_y, p_z\}$$

$$\vec{P} = P \cdot \{\hat{x}, \hat{y}, \hat{z}\} = \{p_x, p_y, p_z\} \cdot \{\hat{x}, \hat{y}, \hat{z}\} = p_x \hat{x} + p_y \hat{y} + p_z \hat{z}$$

Direction Vector: $\vec{P}$

Representation of a vector



$${}^A P = \{p_x, p_y, p_z\} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\vec{{}^A P} = {}^A P \cdot \{\hat{x}, \hat{y}, \hat{z}\} = \{p_x, p_y, p_z\} \cdot \{\hat{x}, \hat{y}, \hat{z}\} = p_x \hat{x} + p_y \hat{y} + p_z \hat{z}$$

$${}^B P' = \{p'_x, p'_y, p'_z\} = \begin{pmatrix} p'_x \\ p'_y \\ p'_z \end{pmatrix}$$

$$\vec{{}^B P'} = {}^B P' \cdot \{\hat{x}', \hat{y}', \hat{z}'\} = \{p'_x, p'_y, p'_z\} \cdot \{\hat{x}', \hat{y}', \hat{z}'\} = p'_x \hat{x}' + p'_y \hat{y}' + p'_z \hat{z}'$$

$$\vec{{}^A P} = \vec{{}^B P'}$$

$$\hat{\mathbf{I}} = \{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

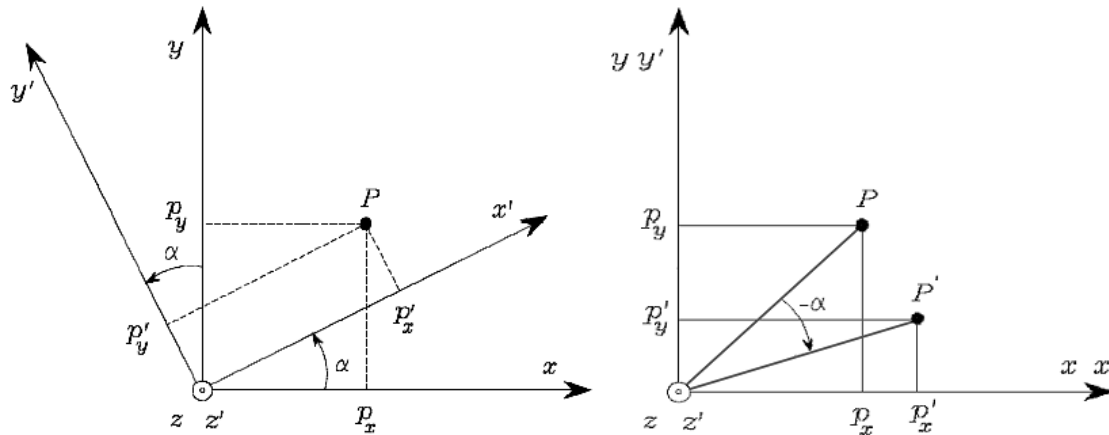
$${}^B\mathbf{P}' = \begin{pmatrix} p_x' \\ p_y' \\ p_z' \end{pmatrix} = {}^B_B\mathbf{R} {}^B\mathbf{P} = (\vec{x'} \ \vec{y'} \ \vec{z'})^T {}^B\mathbf{P}$$

$$= \left(\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \right)^T \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \left(\begin{pmatrix} x_x' & x_y' & x_z' \\ y_x' & y_y' & y_z' \\ z_x' & z_y' & z_z' \end{pmatrix} \cdot \begin{pmatrix} x_x & x_y & x_z \\ y_x & y_y & y_z \\ z_x & z_y & z_z \end{pmatrix} \right)^T \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \vec{x'} \cdot \hat{x} & \vec{y'} \cdot \hat{x} & \vec{z'} \cdot \hat{x} \\ \vec{x'} \cdot \hat{y} & \vec{y'} \cdot \hat{y} & \vec{z'} \cdot \hat{y} \\ \vec{x'} \cdot \hat{z} & \vec{y'} \cdot \hat{z} & \vec{z'} \cdot \hat{z} \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\begin{pmatrix} x_x x_x' + x_y x_y' + x_z x_z' & x_x y_x' + x_y y_y' + x_z y_z' & x_x z_x' + x_y z_y' + x_z z_z' \\ y_x x_x' + y_y x_y' + y_z x_z' & y_x y_x' + y_y y_y' + y_z y_z' & y_x z_x' + y_y z_y' + y_z z_z' \\ z_x x_x' + z_y x_y' + z_z x_z' & z_x y_x' + z_y y_y' + z_z y_z' & z_x z_x' + z_y z_y' + z_z z_z' \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$${}^B\mathbf{P}' = {}^B_B\mathbf{R} {}^B\mathbf{P} = (\vec{x'} \ \vec{y'} \ \vec{z'}) \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \vec{x'} \cdot \hat{x} & \vec{y'} \cdot \hat{x} & \vec{z'} \cdot \hat{x} \\ \vec{x'} \cdot \hat{y} & \vec{y'} \cdot \hat{y} & \vec{z'} \cdot \hat{y} \\ \vec{x'} \cdot \hat{z} & \vec{y'} \cdot \hat{z} & \vec{z'} \cdot \hat{z} \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \vec{x'} \cdot \hat{x} p_x + \vec{y'} \cdot \hat{x} p_y + \vec{z'} \cdot \hat{x} p_z \\ \vec{x'} \cdot \hat{y} p_x + \vec{y'} \cdot \hat{y} p_y + \vec{z'} \cdot \hat{y} p_z \\ \vec{x'} \cdot \hat{z} p_x + \vec{y'} \cdot \hat{z} p_y + \vec{z'} \cdot \hat{z} p_z \end{pmatrix} = \begin{pmatrix} p_x' \\ p_y' \\ p_z' \end{pmatrix}$$

■ Example



$$\begin{pmatrix} p_x' \\ p_y' \\ p_z' \end{pmatrix} = \begin{pmatrix} \vec{x'} \cdot \hat{x} & \vec{y'} \cdot \hat{x} & \vec{z'} \cdot \hat{x} \\ \vec{x'} \cdot \hat{y} & \vec{y'} \cdot \hat{y} & \vec{z'} \cdot \hat{y} \\ \vec{x'} \cdot \hat{z} & \vec{y'} \cdot \hat{z} & \vec{z'} \cdot \hat{z} \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \cos[-\alpha] & \cos[90^\circ - \alpha] & \cos[90^\circ] \\ \cos[90^\circ + \alpha] & \cos[-\alpha] & \cos[90^\circ] \\ \cos[90^\circ] & \cos[90^\circ] & \cos[0^\circ] \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \cos[\alpha] & \sin[\alpha] & 0 \\ -\sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$$p_x' = p_x \cos[\alpha] + p_y \sin[\alpha]$$

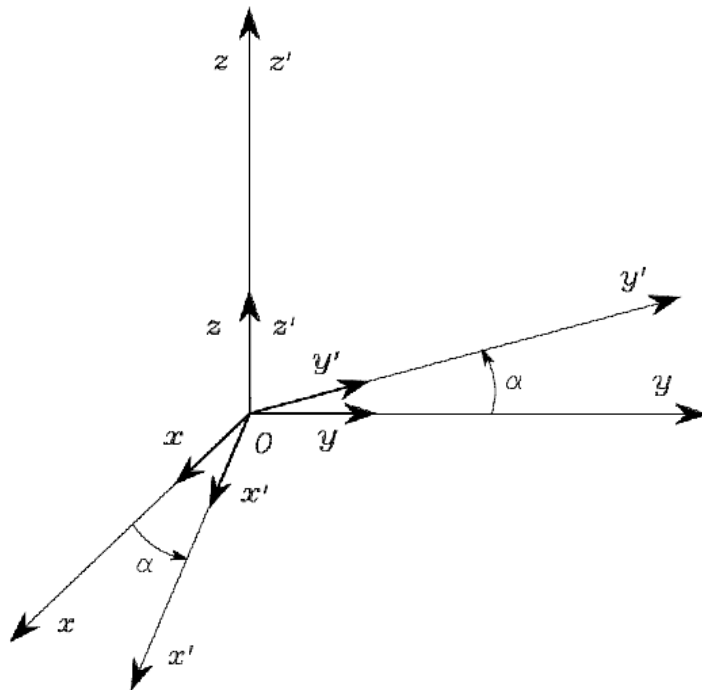
$$p_y' = -p_x \sin[\alpha] + p_y \cos[\alpha]$$

$$p_z' = p_z$$

- 2D Rotation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos[\theta] & -\sin[\theta] \\ \sin[\theta] & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos[\theta] - y \sin[\theta] \\ y \cos[\theta] + x \sin[\theta] \end{pmatrix}$$

- 3D Rotation



- rotation of α about z

$$R_z[\alpha] = \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

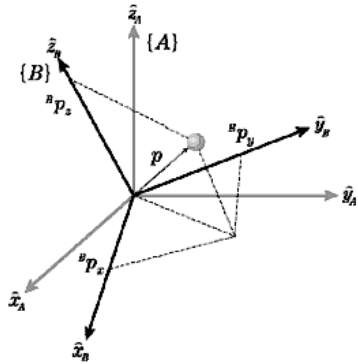
- rotation of β about y

$$R_y[\beta] = \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix}$$

- rotation of γ about x

$$R_x[\gamma] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\gamma] & -\sin[\gamma] \\ 0 & \sin[\gamma] & \cos[\gamma] \end{pmatrix}$$

위치 표현 (Position and Orientation)



좌표계 $\{A\}$, $\{B\}$ 에서 본 점 P의 위치

$\hat{x}_B, \hat{y}_B, \hat{z}_B$ 를 좌표계 $\{A\}$ 의 좌표성분으로 표시하면

$$\hat{x}_B = \hat{x}_B \hat{x}_A + \hat{x}_B \hat{y}_A + \hat{x}_B \hat{z}_A$$

$$\hat{x}_B = \begin{pmatrix} \hat{x}_B \hat{x}_A \\ \hat{x}_B \hat{y}_A \\ \hat{x}_B \hat{z}_A \end{pmatrix}$$

좌표계 $\{A\}$ 에서 본 벡터의 방향, Direction Vector (방향 벡터)

$$P = \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}$$

상대 개념 위치

상대위치표현 ${}^A P, {}^B P$ 의 표현

$${}^A P = \begin{bmatrix} {}^A P_x \\ {}^A P_y \\ {}^A P_z \end{bmatrix}$$

$${}^B P = \begin{bmatrix} {}^B P_x \\ {}^B P_y \\ {}^B P_z \end{bmatrix}$$

$$\vec{P} = {}^B P \cdot \{\hat{x}_B, \hat{y}_B, \hat{z}_B\}$$

$$= \begin{bmatrix} {}^B P_x \\ {}^B P_y \\ {}^B P_z \end{bmatrix} \cdot \{\hat{x}_B, \hat{y}_B, \hat{z}_B\}$$

$$= {}^B P_x \hat{x}_B + {}^B P_y \hat{y}_B + {}^B P_z \hat{z}_B$$

$$= {}^A P_x \hat{x}_A + {}^A P_y \hat{y}_A + {}^A P_z \hat{z}_A$$

$$= \begin{bmatrix} {}^A P_x \\ {}^A P_y \\ {}^A P_z \end{bmatrix} \cdot \{\hat{x}_A, \hat{y}_A, \hat{z}_A\}$$

$$= {}^A P \cdot \{\hat{x}_A, \hat{y}_A, \hat{z}_A\}$$

$$\begin{cases} {}^A P_X, {}^A P_Y, {}^A P_Z : P \text{의 } \hat{x}_A, \hat{y}_A, \hat{z}_A \text{ 방향 성분} \\ {}^B P_X, {}^B P_Y, {}^B P_Z : P \text{의 } \hat{x}_B, \hat{y}_B, \hat{z}_B \text{ 방향 성분} \end{cases}$$

$$\blacksquare {}^B P = {}^A_B R \cdot {}^A P$$

{A} 좌표계 성분을 {B} 좌표계 성분으로 변환

$${}^B P = {}^A_B R \cdot {}^A P$$

$$\begin{pmatrix} {}^B P_X \\ {}^B P_Y \\ {}^B P_Z \end{pmatrix} = \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix} \cdot \begin{pmatrix} {}^A P_X \\ {}^A P_Y \\ {}^A P_Z \end{pmatrix}$$

→ 좌표계 {B} 에 대해 표현된 점P의 위치 ${}^B P$ 가 좌표계 {A} 의 좌표값 ${}^A P$ 로

회전행렬 ${}^A_B R$ 을 통하여 변환

$$\blacksquare \text{Rotation Matrix } {}^A_B R$$

좌표계 {A} 에 대한 좌표계 {B} 의 회전행렬

$$\hat{x}_B = \hat{x}_B \cdot \hat{x}_A + \hat{x}_B \cdot \hat{y}_A + \hat{x}_B \cdot \hat{z}_A$$

$$\hat{y}_B = \hat{y}_B \cdot \hat{x}_A + \hat{y}_B \cdot \hat{y}_A + \hat{y}_B \cdot \hat{z}_A$$

$$\hat{z}_B = \hat{z}_B \cdot \hat{x}_A + \hat{z}_B \cdot \hat{y}_A + \hat{z}_B \cdot \hat{z}_A$$

$${}^A_B R = \begin{pmatrix} \hat{x}_B & \hat{y}_B & \hat{z}_B \end{pmatrix} = \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix}$$

$\hat{x}_B \hat{y}_B \hat{z}_B$ 를 각각 {A} 좌표성분 ($\hat{x}_A \hat{y}_A \hat{z}_A$) 으로 표현한 식들의 조합으로

{A} 에서 본 {B} 의 표현식

좌표계 {A} 와 {B} 사이의 변환 행렬으로 {B} 의 {A} 에 대한 상대관계 또는 상대 회전량

물체의 방위 (상대적 회전량) 를 나타냄

$$\blacksquare \text{회전행렬의 성질 } ({}^B_A R = {}^A_B R^{-1} = {}^A_B R^T)$$

좌표계 {B} 에 대한 좌표계 {A} 의 회전행렬 : ${}^B_A R$

좌표계 {A} 에 대한 좌표계 {B} 의 회전행렬 : ${}^A_B R$

$${}^A_B R = \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix} \iff {}^B_A R = \begin{pmatrix} \hat{x}_A \cdot \hat{x}_B & \hat{y}_A \cdot \hat{x}_B & \hat{z}_A \cdot \hat{x}_B \\ \hat{x}_A \cdot \hat{y}_B & \hat{y}_A \cdot \hat{y}_B & \hat{z}_A \cdot \hat{y}_B \\ \hat{x}_A \cdot \hat{z}_B & \hat{y}_A \cdot \hat{z}_B & \hat{z}_A \cdot \hat{z}_B \end{pmatrix}$$

$${}^B_A R = {}^A_B R^T$$

$$\begin{cases} {}^A P = {}^B R \cdot {}^B P \\ {}^B P = {}^A R \cdot {}^A P \end{cases} \Rightarrow {}^A P = {}^A R \cdot {}^B P = {}^B R \cdot {}^A R \cdot {}^A P \Rightarrow {}^B R \cdot {}^A R = I$$

$${}^B R = {}^A R^{-1} = {}^A R^T$$

3D Orthonormal Basis → Unit Vectors

$$\hat{x}_A, \hat{y}_A, \hat{z}_A$$

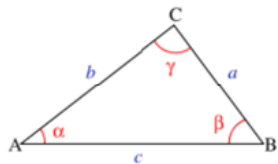
$$\hat{A} = \begin{pmatrix} \hat{x}_{Ax} & \hat{y}_{Ax} & \hat{z}_{Ax} \\ \hat{x}_{Ay} & \hat{y}_{Ay} & \hat{z}_{Ay} \\ \hat{x}_{Az} & \hat{y}_{Az} & \hat{z}_{Az} \end{pmatrix}$$

$$|\hat{x}_A| = |\hat{y}_A| = |\hat{z}_A| = 1$$

$$\hat{x}_A \cdot \hat{y}_A = 0, \hat{y}_A \cdot \hat{z}_A = 0, \hat{z}_A \cdot \hat{x}_A = 0$$

$$\hat{x}_A \times \hat{y}_A = \hat{z}_A$$

Low of Cosines



$$c^2 = a^2 + b^2 - 2 a b \cos [\gamma]$$

$$a^2 = b^2 + c^2 - 2 b c \cos [\alpha]$$

$$b^2 = a^2 + c^2 - 2 a c \cos [\beta]$$

Inner Product Space

$$x \cdot y = |x| |y| \cos [\theta]$$

Half-Angle Identities

$$\cos \left[\frac{\theta}{2} \right] = \sqrt{\frac{1 + \cos [\theta]}{2}}$$

$$\sin \left[\frac{\theta}{2} \right] = \sqrt{\frac{1 - \cos [\theta]}{2}}$$

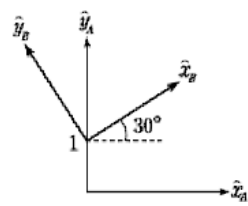
$$\tan [\theta] = \frac{\sin [\theta]}{\cos [\theta]}$$

$$\text{Sec}[\theta] = \frac{1}{\text{Cos}[\theta]}$$

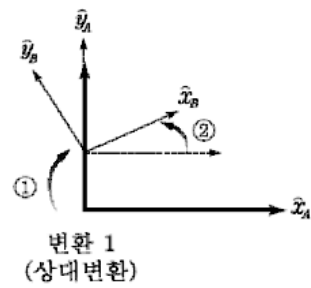
연속좌표 변환: 절대변환과 상대변환

- 이동좌표계 (moving frame)
 - 변환 (회전 또는 병진) 에 의해 새롭게 얻어지는 좌표계
- 고정좌표계 (fixed frame)
 - 변환에 관계없이 처음과 동일하게 고정된 좌표계
- 절대변환 (absolute transform)
 - 전 단계의 변환에 관계없이 후속변환을 계속 초기의 고정된 기준좌표인 고정좌표계에 대해 수행하는 것
- 상대변환 (relative transform)
 - 각 단계의 변환 후 얻어진 새로운 좌표계인 이동좌표계에 대하여 다음 변환이 수행 되는 것
- 좌표계 {A}와 {B} 사이의 변환을 나타내는 동차변환행렬

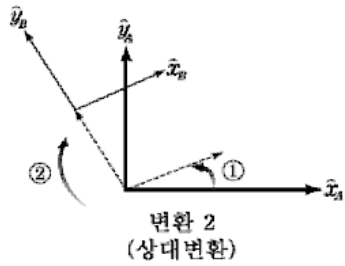
$${}^A_B\mathbf{T} = \text{Trans}[\text{p}_x, \text{p}_y, \text{p}_z] \cdot \text{Rot}[\text{z}_A, \alpha]$$



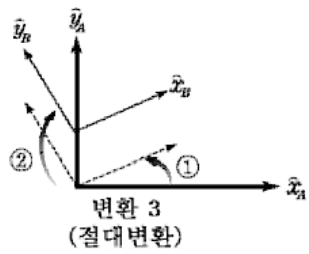
- O: 이동좌표계를 따라 병진 후 이동좌표계에 대해 회전



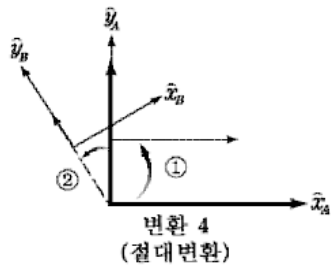
- X: 이동좌표계를 따라 회전 후 이동좌표계에 대해 병진



- O: 고정좌표계를 따라 회전 후 고정좌표계에 대해 병진



- X: 고정좌표계를 따라 병진 후 고정좌표계에 대해 회전



- 올바른 변환을 나타내는 조합

상대변환 : 이동좌표계에 대한 병진 → 이동좌표계에 대해 회전

절대변환 : 고정좌표계에 대한 회전 → 고정좌표계에 대해 병진

두가지 모두 동일한 결과 ${}^A_BT$ 를 나타냄

$${}^A_BT = \text{Trans}[p_x, p_y, p_z] \cdot \text{Rot}[\alpha, \beta, \gamma]$$

∴ 상대변환의 구현 (post-multiply)

회전과 병진을 나타내는 변환행렬이 앞에서부터 뒤로 순차적으로 곱해져야 함

∴ 절대변환의 구현 (pre-multiply)

반대로 해당 변환행렬이 뒤에서부터 앞으로 순차적으로 곱해져야 함

$${}^A\mathbf{r}_D = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$${}^B\mathbf{r}_D = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = {}^B\mathbf{r}_D'$$

$${}^A\mathbf{r}_{D'} = {}^A\mathbf{T}_B \cdot {}^B\mathbf{r}_D'$$

$${}^A\mathbf{r}_{C'} = {}^A\mathbf{T}_B \cdot {}^B\mathbf{r}_{C'}$$

방위의 다른 표현

- 방위의 표현

회전행렬 (3×3 행렬) → 다른 방위를 지정할 때 마다 9 개의 원소를 지정해 주어야 함.

작은 수의 입력으로 방위를 표시할 수 있다면, 보다 효과적

- EULER ANGLES

- rotation matrix

9 parameters with 6 constraints

- minimal representation of orientation

3 independent parameters

- Euler angle과 Roll-Pitch-Yaw 방법

물체의 방위를 연속된 세번의 회전만으로 표현할 수 있어 회전행렬 보다 효율적으로 방위를 표현하는 방법

Euler angle : 초기 방위를 나타내는 좌표계를 이동좌표계의 주축에 대해 연속적으로 세번 회전을 수행하여 특정 방위를 얻어내는 방법

Roll - Pitch - Yaw : 고정기준좌표계에 대해 연속적으로 세번 회전을하여 원하는 방위를 표시하는 방법

초기 좌표를 세번만 회전하므로 매 회전시의 회전량 3 개만 정의하면 되므로 9 개를 입력해야 하는 회전행렬에 비해 효율적이다.

미니멀레이터 팔단부를 3 자 유도로 구성함으로써 모든 방위를 구현할 수 있게 되는 근거

각각 12 가지 표현법 존재

- Z-Y-Z 오일러 각

좌표계 {A} 의 Z축에 대해 α 만큼 회전 후

이동 좌표계 즉 회전 후 얻어진 새로운 좌표계 {B'}의 Y축에 대해 β 만큼 회전한 후,

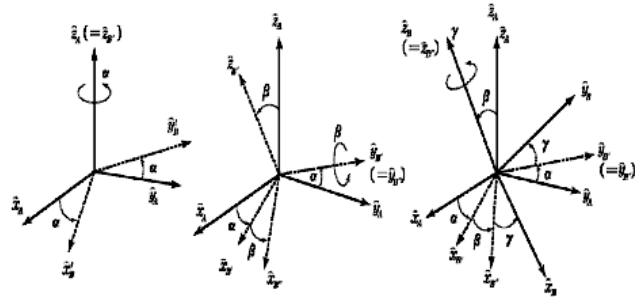
다시 새로운 얻어진 좌표계 (이동좌표계) {B''}의 Z축에 대해 γ 만큼 연속적으로 회전을 하여 특정방위를 나타내는 좌표계 (B)를 얻는 방법

모든 회전이 이동 좌표계에 대한 회전을 이용한 상대 변환이므로 해당 변환행렬을 **앞에서부터 순차적**으로 곱한다.

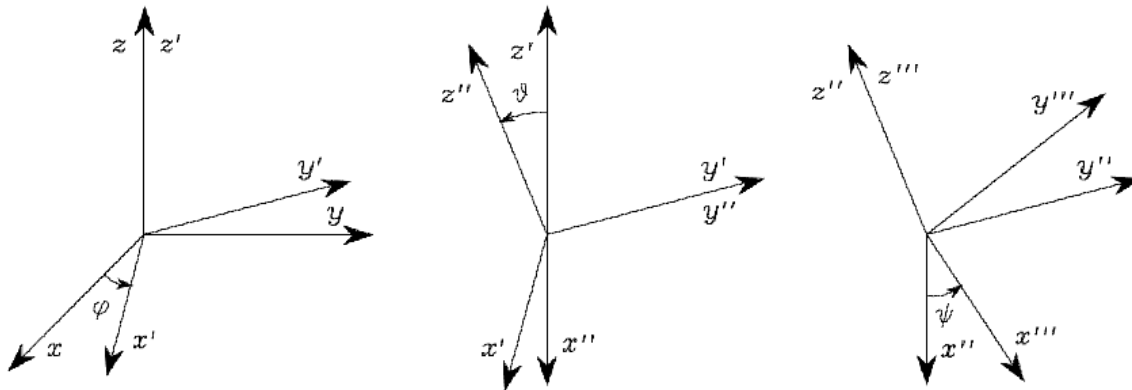
$${}^A_B T = R_z[\alpha] R_y[\beta] R_x[\gamma]$$

$$= \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[\gamma] & -\sin[\gamma] & 0 \\ \sin[\gamma] & \cos[\gamma] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos[\alpha] \cos[\beta] \cos[\gamma] - \sin[\alpha] \sin[\gamma] & -\cos[\gamma] \sin[\alpha] - \cos[\alpha] \cos[\beta] \sin[\gamma] & \cos[\alpha] \sin[\beta] \\ \cos[\beta] \cos[\gamma] \sin[\alpha] + \cos[\alpha] \sin[\gamma] & \cos[\alpha] \cos[\gamma] - \cos[\beta] \sin[\alpha] \sin[\gamma] & \sin[\alpha] \sin[\beta] \\ -\cos[\gamma] \sin[\beta] & \sin[\beta] \sin[\gamma] & \cos[\beta] \end{pmatrix}$$



■ ZYZ angles



$$R[\phi] = R_z[\phi] R_y[\theta] R_x[\psi]$$

$$= \begin{pmatrix} \cos[\phi] & -\sin[\phi] & 0 \\ \sin[\phi] & \cos[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} // \text{MatrixForm}$$

$$= \begin{pmatrix} \cos[\theta] \cos[\varphi] \cos[\psi] - \sin[\varphi] \sin[\psi] & -\cos[\psi] \sin[\varphi] - \cos[\theta] \cos[\varphi] \sin[\psi] & \cos[\varphi] \sin[\theta] \\ \cos[\theta] \cos[\psi] \sin[\varphi] + \cos[\varphi] \sin[\psi] & \cos[\varphi] \cos[\psi] - \cos[\theta] \sin[\varphi] \sin[\psi] & \sin[\theta] \sin[\varphi] \\ -\cos[\psi] \sin[\theta] & \sin[\theta] \sin[\psi] & \cos[\theta] \end{pmatrix}$$

■ Inverse problem

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

the three ZYZ angles are $(\theta \in (0, \pi))$

$$\varphi = \text{ArcTan}[r_{13}, r_{23}]$$

$$\theta = \text{ArcTan}[r_{33}, \sqrt{r_{13}^2 + r_{23}^2}]$$

$$\psi = \text{ArcTan}[-r_{31}, r_{32}]$$

or $(\theta \in (-\pi, 0))$

$$\varphi = \text{ArcTan}[-r_{13}, -r_{23}]$$

$$\theta = \text{ArcTan}[r_{33}, -\sqrt{r_{13}^2 + r_{23}^2}]$$

$$\psi = \text{ArcTan}[r_{31}, -r_{32}]$$

■ Z-Y-X 오일러 각 (이동 좌표계)

좌표계 {A}의 Z축에 대해 α 만큼 회전 후

이동좌표계 즉 회전후 얻어진 새로운 좌표계 {B'}의 Y축에 대해 β 만큼 회전한 후,

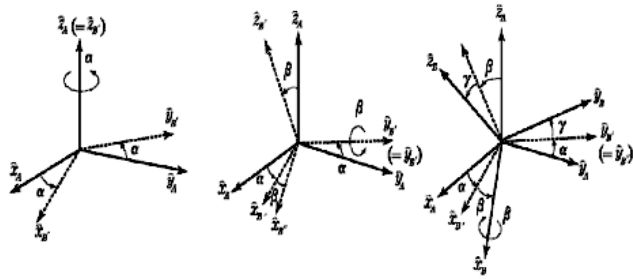
다시 새로운 얻어진 좌표계 (이동좌표계) {B'}의 X축에 대해 γ 만큼 연속적으로 회전을 하여 특정방위를 나타내는 좌표계 {B}를 얻는 방법

모든 회전이 이동 좌표계에 대한 회전을 이용한 상대변환이므로 해당 변환행렬을 **앞에서부터 순차적**으로 곱한다.

$${}^A_B T = R_z[\alpha] R_y[\beta] R_x[\gamma]$$

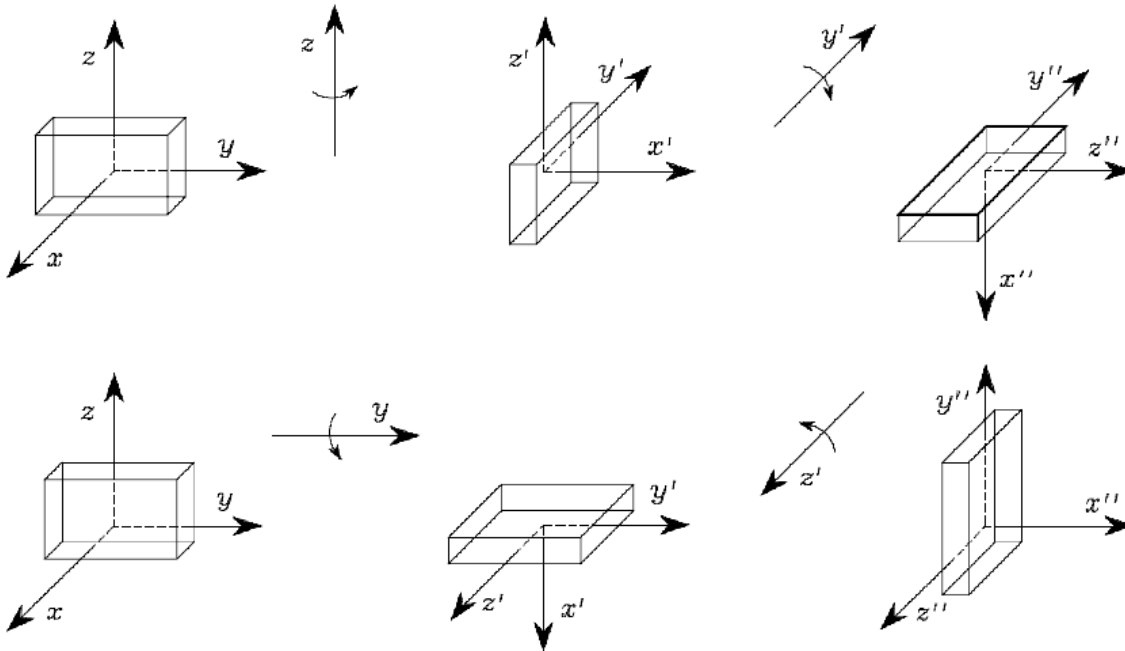
$$= \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\gamma] & -\sin[\gamma] \\ 0 & \sin[\gamma] & \cos[\gamma] \end{pmatrix}$$

$$= \begin{pmatrix} \cos[\alpha] \cos[\beta] & -\cos[\gamma] \sin[\alpha] + \cos[\alpha] \sin[\beta] \sin[\gamma] & \cos[\alpha] \cos[\gamma] \sin[\beta] + \sin[\alpha] \sin[\gamma] \\ \cos[\beta] \sin[\alpha] & \cos[\alpha] \cos[\gamma] + \sin[\alpha] \sin[\beta] \sin[\gamma] & \cos[\gamma] \sin[\alpha] \sin[\beta] - \cos[\alpha] \sin[\gamma] \\ -\sin[\beta] & \cos[\beta] \sin[\gamma] & \cos[\beta] \cos[\gamma] \end{pmatrix}$$



■ Current frame rotation

$$R_2^0 = R_1^0 R_2^1$$



■ Roll-Pitch-Yaw (고정 좌표계)

고정기준좌표계 {A}의 X축에 대해 γ 만큼 회전 후

고정좌표계의 Y축에 대해 β 만큼 회전한 후,

고정좌표계의 Z축에 대해 α 만큼 연속적으로 회전을 하여 특정방위를 나타내는 좌표계 {B}를 얻는 방법

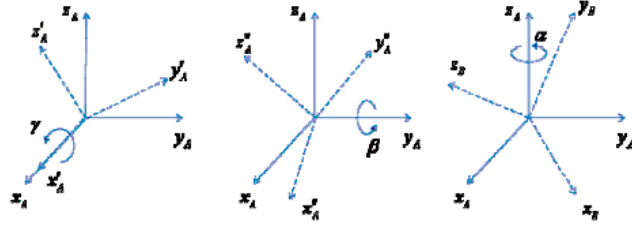
모든 회전이 고정 기준 좌표계에 대한 회전을 이용한 절대변환이므로 해당 변환행렬을 **뒤에서부터 앞으로 역순**으로 곱한다.

Z - Y - X 오일러와 동일

$${}^A_B T = R_z[\alpha] R_y[\beta] R_x[\gamma]$$

$$= \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\gamma] & -\sin[\gamma] \\ 0 & \sin[\gamma] & \cos[\gamma] \end{pmatrix}$$

$$= \begin{pmatrix} \cos[\alpha] \cos[\beta] & -\cos[\gamma] \sin[\alpha] + \cos[\alpha] \sin[\beta] \sin[\gamma] & \cos[\alpha] \cos[\gamma] \sin[\beta] + \sin[\alpha] \sin[\gamma] \\ \cos[\beta] \sin[\alpha] & \cos[\alpha] \cos[\gamma] + \sin[\alpha] \sin[\beta] \sin[\gamma] & \cos[\gamma] \sin[\alpha] \sin[\beta] - \cos[\alpha] \sin[\gamma] \\ -\sin[\beta] & \cos[\beta] \sin[\gamma] & \cos[\beta] \cos[\gamma] \end{pmatrix}$$



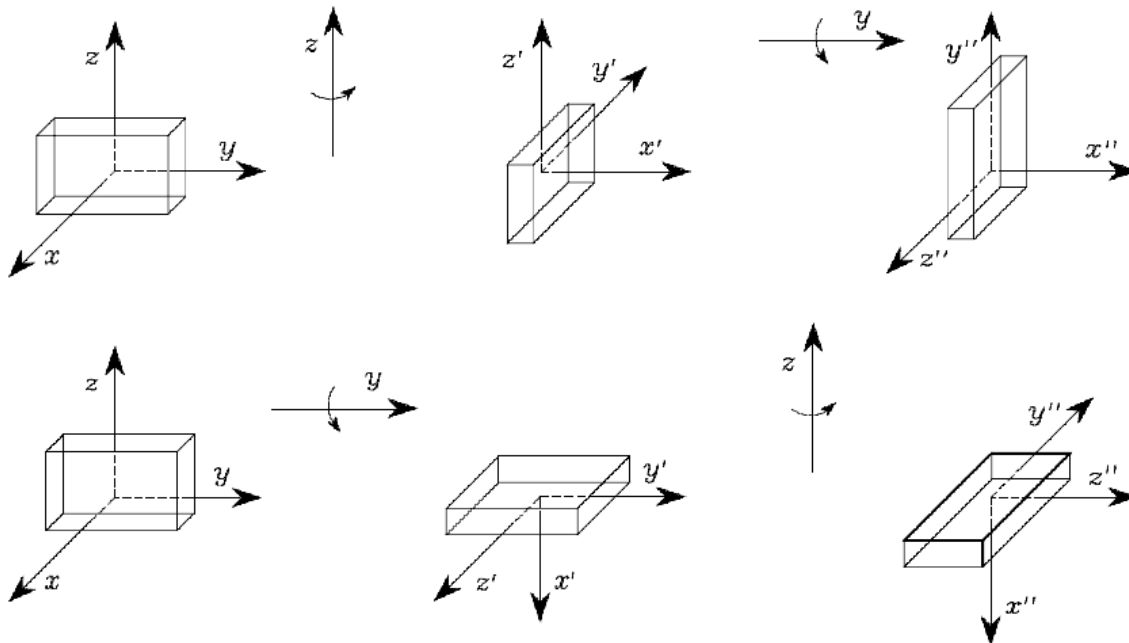
요

피치

롤

Fixed frame rotation

$$R_2^0 = R_2^1 R_1^0$$



Rotation Matrix ${}^A_B\mathbf{R}$

Rotation Matrix

Direction Cosine Matrix

- 좌표계 {A}에 대한 좌표계 {B}의 회전행렬

$$\hat{x}_B = \hat{x}_B \hat{x}_A + \hat{x}_B \hat{y}_A + \hat{x}_B \hat{z}_A$$

$$\hat{y}_B = \hat{y}_B \hat{x}_A + \hat{y}_B \hat{y}_A + \hat{y}_B \hat{z}_A$$

$$\hat{z}_B = \hat{z}_B \hat{x}_A + \hat{z}_B \hat{y}_A + \hat{z}_B \hat{z}_A$$

$${}^A_B\mathbf{R} = \begin{pmatrix} \hat{x}_B & \hat{y}_B & \hat{z}_B \end{pmatrix} = \begin{pmatrix} \hat{x}_B \hat{x}_A & \hat{y}_B \hat{x}_A & \hat{z}_B \hat{x}_A \\ \hat{x}_B \hat{y}_A & \hat{y}_B \hat{y}_A & \hat{z}_B \hat{y}_A \\ \hat{x}_B \hat{z}_A & \hat{y}_B \hat{z}_A & \hat{z}_B \hat{z}_A \end{pmatrix}$$

$\hat{x}_B \ \hat{y}_B \ \hat{z}_B$ 를 각각 {A} 좌표성분 ($\hat{x}_A \ \hat{y}_A \ \hat{z}_A$)으로 표현한 식들의 조합으로 {A}에서 본 {B}의 표현식

좌표계 {A}와 {B} 사이의 변환행렬으로 {B}의 {A}에 대한 상대관계 또는 상대회전량

물체의 방위 (상대적 회전량)를 나타냄

- ${}^A\mathbf{P} = {}^A_B\mathbf{R} {}^B\mathbf{P}$

{B} 좌표계 성분을 {A} 좌표계 성분으로 변환

$${}^A\mathbf{P} = \begin{pmatrix} {}^A\mathbf{p}_x \\ {}^A\mathbf{p}_y \\ {}^A\mathbf{p}_z \end{pmatrix} = \begin{pmatrix} {}^A\mathbf{P} \cdot \hat{x}_A \\ {}^A\mathbf{P} \cdot \hat{y}_A \\ {}^A\mathbf{P} \cdot \hat{z}_A \end{pmatrix} = \begin{pmatrix} \mathbf{P} \cdot \hat{x}_A \\ \mathbf{P} \cdot \hat{y}_A \\ \mathbf{P} \cdot \hat{z}_A \end{pmatrix} = \begin{pmatrix} {}^B\mathbf{P} \cdot \hat{x}_A \\ {}^B\mathbf{P} \cdot \hat{y}_A \\ {}^B\mathbf{P} \cdot \hat{z}_A \end{pmatrix} = \begin{pmatrix} ({}^B\mathbf{p}_x \hat{x}_B + {}^B\mathbf{p}_y \hat{y}_B + {}^B\mathbf{p}_z \hat{z}_B) \cdot \hat{x}_A \\ ({}^B\mathbf{p}_x \hat{x}_B + {}^B\mathbf{p}_y \hat{y}_B + {}^B\mathbf{p}_z \hat{z}_B) \cdot \hat{y}_A \\ ({}^B\mathbf{p}_x \hat{x}_B + {}^B\mathbf{p}_y \hat{y}_B + {}^B\mathbf{p}_z \hat{z}_B) \cdot \hat{z}_A \end{pmatrix}$$

$$= \begin{pmatrix} \hat{x}_B \hat{x}_A & \hat{y}_B \hat{x}_A & \hat{z}_B \hat{x}_A \\ \hat{x}_B \hat{y}_A & \hat{y}_B \hat{y}_A & \hat{z}_B \hat{y}_A \\ \hat{x}_B \hat{z}_A & \hat{y}_B \hat{z}_A & \hat{z}_B \hat{z}_A \end{pmatrix} \cdot \begin{pmatrix} {}^B\mathbf{p}_x \\ {}^B\mathbf{p}_y \\ {}^B\mathbf{p}_z \end{pmatrix}$$

$${}^A\mathbf{P} = {}^A_B\mathbf{R} {}^B\mathbf{P}$$

→ 좌표계 {B}에 대해 표현된 점P의 위치 ${}^B\mathbf{P}$ 가 좌표계 {A}의 좌표값 ${}^A\mathbf{P}$ 로 회전전행렬 ${}^A_B\mathbf{R}$ 을 통하여 변환됨을 의미

⇒ ${}^A_B\mathbf{R}$: Transformation Matrix (변환행렬)라고 정의, 좌표계 {A}에 대한 좌표계 {B}의 회전행렬

- 회전행렬의 성질 (${}^B_A\mathbf{R} = {}^A_B\mathbf{R}^{-1} = {}^A_B\mathbf{R}^T$)

좌표계 {B}에 대한 좌표계 {A}의 회전행렬 : ${}^B_A\mathbf{R}$

좌표계 {A}에 대한 좌표계 {B}의 회전행렬 : ${}^A_B\mathbf{R}$

$${}^B_R = \begin{pmatrix} \hat{x}_A & \hat{x}_B & \hat{y}_A & \hat{x}_B & \hat{z}_A & \hat{x}_B \\ \hat{x}_A & \hat{y}_B & \hat{y}_A & \hat{y}_B & \hat{z}_A & \hat{y}_B \\ \hat{x}_A & \hat{z}_B & \hat{y}_A & \hat{z}_B & \hat{z}_A & \hat{z}_B \end{pmatrix} \iff {}^A_B R = \begin{pmatrix} \hat{x}_B & \hat{x}_A & \hat{y}_B & \hat{x}_A & \hat{z}_B & \hat{x}_A \\ \hat{x}_B & \hat{y}_A & \hat{y}_B & \hat{y}_A & \hat{z}_B & \hat{y}_A \\ \hat{x}_B & \hat{z}_A & \hat{y}_B & \hat{z}_A & \hat{z}_B & \hat{z}_A \end{pmatrix}$$

$${}^B_R R = {}^A_B R^T$$

$$\begin{cases} {}^A_P = {}^A_B R {}^B_P \\ {}^B_P = {}^B_A R {}^A_P \end{cases} \implies {}^A_P = {}^A_B R {}^B_P = {}^A_B R {}^B_A R {}^A_P \implies {}^A_B R {}^B_A R = I$$

$${}^B_A R = {}^A_B R^{-1} = {}^A_B R^T$$

- Cartesian coordinate (직교좌표계)

$$|\hat{x}_A| = |\hat{x}_B| = 1$$

- Inner Product Space

$$\hat{x}_B \cdot \hat{x}_A = |\hat{x}_B| |\hat{x}_A| \cos[\gamma]$$

- Law of Cosines

$$|\overrightarrow{\hat{x}_A \hat{x}_B}|^2 = |\hat{x}_A|^2 + |\hat{x}_B|^2 - 2 |\hat{x}_A| |\hat{x}_B| \cos[\gamma]$$

$$\hat{x}_B \cdot \hat{x}_A = |\hat{x}_B| |\hat{x}_A| \cos[\gamma]$$

$$\cos[\gamma] = 1 - \frac{(x_{Bx} - x_{Ax})^2 + (x_{By} - x_{Ay})^2 + (x_{Bz} - x_{Az})^2}{2}$$

$$\hat{x}_B \cdot \hat{x}_A = 1 - \frac{(x_{Bx} - x_{Ax})^2 + (x_{By} - x_{Ay})^2 + (x_{Bz} - x_{Az})^2}{2}$$

- ${}^A_B R$: Transformation Matrix (변환행렬)

$$\begin{pmatrix} \hat{x}_B & \hat{x}_A & \hat{y}_B & \hat{x}_A & \hat{z}_B & \hat{x}_A \\ \hat{x}_B & \hat{y}_A & \hat{y}_B & \hat{y}_A & \hat{z}_B & \hat{y}_A \\ \hat{x}_B & \hat{z}_A & \hat{y}_B & \hat{z}_A & \hat{z}_B & \hat{z}_A \end{pmatrix} =$$

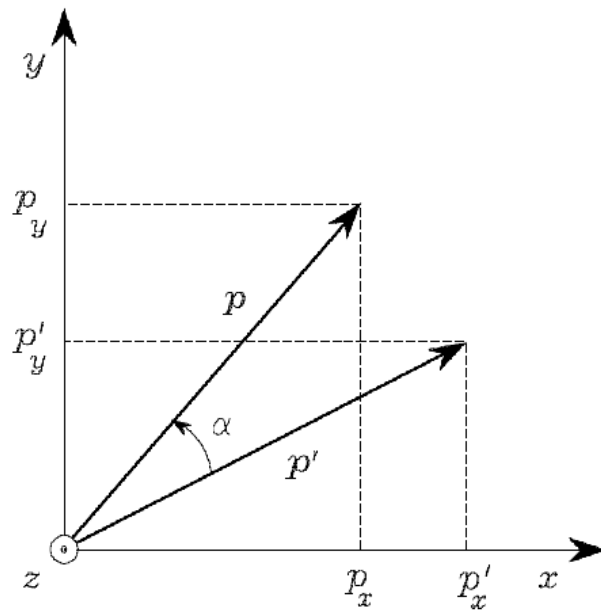
$$\begin{pmatrix} 1 - \frac{(x_{Bx}-x_{Ax})^2 + (x_{By}-x_{Ay})^2 + (x_{Bz}-x_{Az})^2}{2} & 1 - \frac{(y_{Bx}-x_{Ax})^2 + (y_{By}-x_{Ay})^2 + (y_{Bz}-x_{Az})^2}{2} & 1 - \frac{(z_{Bx}-x_{Ax})^2 + (z_{By}-x_{Ay})^2 + (z_{Bz}-x_{Az})^2}{2} \\ 1 - \frac{(x_{Bx}-y_{Ax})^2 + (x_{By}-y_{Ay})^2 + (x_{Bz}-y_{Az})^2}{2} & 1 - \frac{(y_{Bx}-y_{Ax})^2 + (y_{By}-y_{Ay})^2 + (y_{Bz}-y_{Az})^2}{2} & 1 - \frac{(z_{Bx}-y_{Ax})^2 + (z_{By}-y_{Ay})^2 + (z_{Bz}-y_{Az})^2}{2} \\ 1 - \frac{(x_{Bx}-z_{Ax})^2 + (x_{By}-z_{Ay})^2 + (x_{Bz}-z_{Az})^2}{2} & 1 - \frac{(y_{Bx}-z_{Ax})^2 + (y_{By}-z_{Ay})^2 + (y_{Bz}-z_{Az})^2}{2} & 1 - \frac{(z_{Bx}-z_{Ax})^2 + (z_{By}-z_{Ay})^2 + (z_{Bz}-z_{Az})^2}{2} \end{pmatrix}$$

Rotation of a vector

$$p = R p'$$

$$p^T p = p'^T R^T R p'$$

- Example



$$p_x = p'_x \cos[\alpha] - p'_y \sin[\alpha]$$

$$p_y = p'_x \sin[\alpha] + p'_y \cos[\alpha]$$

$$p_z = p'_z$$

$$p = R_\alpha[\alpha] \, p'$$

■ Composition Of Rotation Matrices

it describes the mutual orientation between two coordinate

frames; its column vectors are the direction cosines of the

axes of the rotated frame with respect to the original frame

it represents the coordinate transformation between the

coordinates of a point expressed in two different frames

(with common origin)

it is the operator that allows the rotation of a vector in the

same coordinate frame

■ Composition Of Rotation Matrices

$$\mathbf{p}^1 = \mathbf{R}_2^1 \mathbf{p}^2$$

$$\mathbf{p}^0 = \mathbf{R}_1^0 \mathbf{p}^1$$

$$\mathbf{p}^0 = \mathbf{R}_2^0 \mathbf{p}^2$$

$$\mathbf{R}_i^j = \left(\mathbf{R}_j^i \right)^{-1} = \left(\mathbf{R}_j^i \right)^T$$

■ Mathematica

$$\mathbf{R}_\mathbf{x}[\phi_]:= \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \text{Cos}[\phi] & -\text{Sin}[\phi] \\ 0 & \text{Sin}[\phi] & \text{Cos}[\phi] \end{array} \right) // \text{MatrixForm}$$

$$\mathbf{R}_\mathbf{Y}[\theta_]:= \left(\begin{array}{ccc} \text{Cos}[\theta] & 0 & \text{Sin}[\theta] \\ 0 & 1 & 0 \\ -\text{Sin}[\theta] & 0 & \text{Cos}[\theta] \end{array} \right) // \text{MatrixForm}$$

$$\mathbf{R}_\mathbf{z}[\psi_]:= \left(\begin{array}{ccc} \text{Cos}[\psi] & -\text{Sin}[\psi] & 0 \\ \text{Sin}[\psi] & \text{Cos}[\psi] & 0 \\ 0 & 0 & 1 \end{array} \right) // \text{MatrixForm}$$

$$\mathbf{R}_\mathbf{z}[\psi] . \mathbf{R}_\mathbf{Y}[\theta] . \mathbf{R}_\mathbf{x}[\phi]$$

$$= \left(\begin{array}{ccc} \text{Cos}[\psi] & -\text{Sin}[\psi] & 0 \\ \text{Sin}[\psi] & \text{Cos}[\psi] & 0 \\ 0 & 0 & 1 \end{array} \right) . \left(\begin{array}{ccc} \text{Cos}[\theta] & 0 & \text{Sin}[\theta] \\ 0 & 1 & 0 \\ -\text{Sin}[\theta] & 0 & \text{Cos}[\theta] \end{array} \right) . \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \text{Cos}[\phi] & -\text{Sin}[\phi] \\ 0 & \text{Sin}[\phi] & \text{Cos}[\phi] \end{array} \right) // \text{MatrixForm}$$

$$= \left(\begin{array}{ccc} \text{Cos}[\theta] \text{Cos}[\psi] & \text{Cos}[\psi] \text{Sin}[\theta] \text{Sin}[\phi] - \text{Cos}[\phi] \text{Sin}[\psi] & \text{Cos}[\phi] \text{Cos}[\psi] \text{Sin}[\theta] + \text{Sin}[\phi] \text{Sin}[\psi] \\ \text{Cos}[\theta] \text{Sin}[\psi] & \text{Cos}[\phi] \text{Cos}[\psi] + \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi] & -\text{Cos}[\psi] \text{Sin}[\phi] + \text{Cos}[\phi] \text{Sin}[\theta] \text{Sin}[\psi] \\ -\text{Sin}[\theta] & \text{Cos}[\theta] \text{Sin}[\phi] & \text{Cos}[\theta] \text{Cos}[\phi] \end{array} \right)$$

$$\mathbf{R} = \mathbf{R}_\mathbf{x}[\psi] . \mathbf{R}_\mathbf{Y}[\theta] . \mathbf{R}_\mathbf{x}[\phi] = \left(\begin{array}{ccc} \text{Cos}[\theta] \text{Cos}[\psi] & \text{Cos}[\psi] \text{Sin}[\theta] \text{Sin}[\phi] - \text{Cos}[\phi] \text{Sin}[\psi] & \text{Cos}[\phi] \text{Cos}[\psi] \text{Sin}[\theta] + \text{Sin}[\phi] \text{Sin}[\psi] \\ \text{Cos}[\theta] \text{Sin}[\psi] & \text{Cos}[\phi] \text{Cos}[\psi] + \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi] & -\text{Cos}[\psi] \text{Sin}[\phi] + \text{Cos}[\phi] \text{Sin}[\theta] \text{Sin}[\psi] \\ -\text{Sin}[\theta] & \text{Cos}[\theta] \text{Sin}[\phi] & \text{Cos}[\theta] \text{Cos}[\phi] \end{array} \right)$$

Rotation Matrix and Small Angels

$$\text{Cos}[\alpha] \approx 1, \text{Sin}[\alpha] \approx \alpha$$

$$\mathbf{R}_{\mathbf{zyx}} = \mathbf{R}_\mathbf{z}[\psi] \mathbf{R}_\mathbf{Y}[\theta] \mathbf{R}_\mathbf{x}[\phi]$$

$$\left(\begin{array}{ccc} 1 & -\psi & 0 \\ \psi & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) . \left(\begin{array}{ccc} 1 & 0 & \theta \\ 0 & 1 & 0 \\ -\theta & 0 & 1 \end{array} \right) . \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -\phi \\ 0 & \phi & 1 \end{array} \right) // \text{MatrixForm}$$

$$\mathbf{R}_{\mathbf{zyx}} = \left(\begin{array}{ccc} 1 & \theta \phi - \psi & \theta + \phi \psi \\ \psi & 1 + \theta \phi \psi & -\phi + \theta \psi \\ -\theta & \phi & 1 \end{array} \right) \approx \left(\begin{array}{ccc} 1 & -\phi & \theta \\ \phi & 1 & -\psi \\ -\theta & \psi & 1 \end{array} \right)$$

$$\mathbf{R}_{\mathbf{zyx}} = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -\phi \\ 0 & \phi & 1 \end{array} \right) . \left(\begin{array}{ccc} 1 & 0 & \theta \\ 0 & 1 & 0 \\ -\theta & 0 & 1 \end{array} \right) . \left(\begin{array}{ccc} 1 & -\psi & 0 \\ \psi & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) // \text{MatrixForm}$$

$$R_{xyz} = \begin{pmatrix} 1 & -\psi & \theta \\ \theta \phi + \psi & 1 - \theta \phi \psi & -\phi \\ -\theta + \phi \psi & \phi + \theta \psi & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\psi & \theta \\ \psi & 1 & -\phi \\ -\theta & \phi & 1 \end{pmatrix}$$

- **Roll: ϕ**
Pitch: θ

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} + g \begin{pmatrix} \sin[\theta] \\ -\cos[\theta] \sin[\phi] \\ -\cos[\theta] \cos[\phi] \end{pmatrix}$$

- 계산

$$\begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] & \sin[\theta] \sin[\phi] & \cos[\phi] \sin[\theta] \\ 0 & \cos[\phi] & -\sin[\phi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$(0 \ 0 \ -g) \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$(g \sin[\theta] \ -g \cos[\theta] \sin[\phi] \ -g \cos[\theta] \cos[\phi])$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[-\phi] & -\sin[-\phi] \\ 0 & \sin[-\phi] & \cos[-\phi] \end{pmatrix} \cdot \begin{pmatrix} \cos[-\theta] & 0 & \sin[-\theta] \\ 0 & 1 & 0 \\ -\sin[-\theta] & 0 & \cos[-\theta] \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} g \sin[\theta] \\ -g \cos[\theta] \sin[\phi] \\ -g \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[-\phi] & -\sin[-\phi] \\ 0 & \sin[-\phi] & \cos[-\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & \sin[\phi] \\ 0 & -\sin[\phi] & \cos[\phi] \end{pmatrix}$$

- **yaw: ϕ**

$$\mathbf{x} = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}$$

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} = \begin{pmatrix} 1 & \sin[\phi] \tan[\theta] & \cos[\phi] \tan[\theta] \\ 0 & \cos[\phi] & -\sin[\theta] \\ 0 & \sin[\phi] \sec[\theta] & \cos[\phi] \sec[\theta] \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

- **rotation vector: Roll: ϕ , Pitch: θ , yaw: ψ**

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} + g \begin{pmatrix} \sin[\theta] \\ -\cos[\theta] \sin[\phi] \\ -\cos[\theta] \cos[\phi] \end{pmatrix}$$

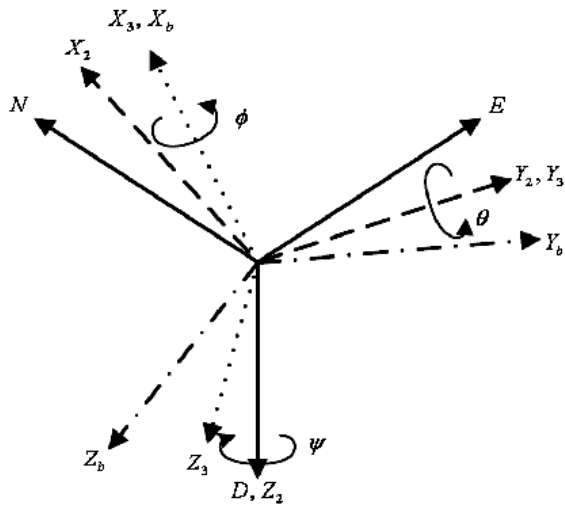
$$(0 \ 0 \ -g) \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$x = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} = \begin{pmatrix} 1 & \sin[\phi] \tan[\theta] & \cos[\phi] \tan[\theta] \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] \sec[\theta] & \cos[\phi] \sec[\theta] \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\tan[\theta] = \frac{\sin[\theta]}{\cos[\theta]}$$

$$\sec[\theta] = \frac{1}{\cos[\theta]}$$

Euler Angle



$$C_n^b = C_3^b[\phi] \ C_2^3[\theta] \ C_n^2[\psi]$$

$$R_{xyz} = R_x[\psi] \ R_y[\theta] \ R_x[\phi]$$

$$= \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$= \begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \phi \\ 0 \\ 0 \end{pmatrix} + C_3^b[\phi] \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + C_3^b[\phi] \ C_2^3[\theta] \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin[\phi] \tan[\theta] & \cos[\phi] \tan[\theta] \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] \sec[\theta] & \cos[\phi] \sec[\theta] \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

■ Derivation

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

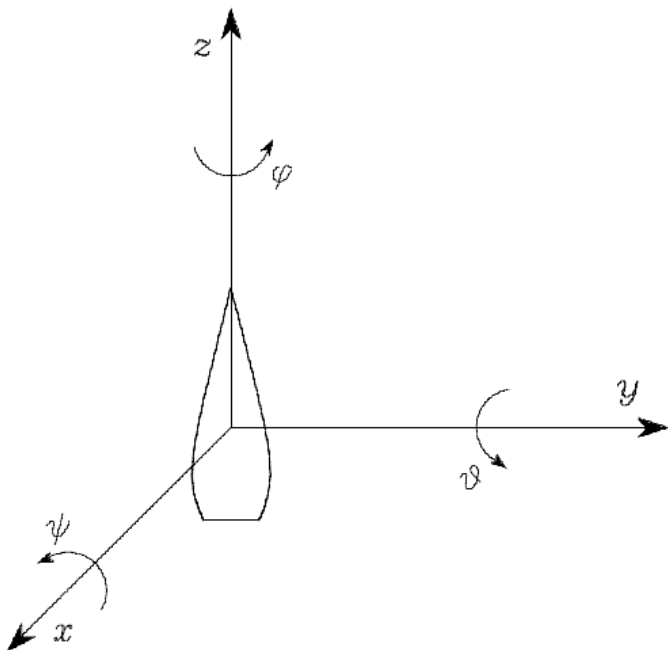
$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot{\phi} + \dot{\psi} \sin[\theta] \\ \cos[\phi] \dot{\theta} - \cos[\theta] \dot{\psi} \sin[\phi] \\ \cos[\theta] \cos[\phi] \dot{\psi} + \dot{\theta} \sin[\phi] \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin[\phi] \tan[\theta] & \cos[\phi] \tan[\theta] \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] \sec[\theta] & \cos[\phi] \sec[\theta] \end{pmatrix} \cdot \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \omega_x + \sin[\phi] \omega_y \tan[\theta] + \cos[\phi] \omega_z \tan[\theta] \\ \cos[\phi] \omega_y - \sin[\phi] \omega_z \\ \sec[\theta] \sin[\phi] \omega_y + \cos[\phi] \sec[\theta] \omega_z \end{pmatrix}$$

$$\text{Solve} \left[\left\{ \dot{\phi} == \omega_x + \sin[\phi] \omega_y \tan[\theta] + \cos[\phi] \omega_z \tan[\theta], \dot{\theta} == \cos[\phi] \omega_y - \sin[\phi] \omega_z, \dot{\psi} == \sec[\theta] \sin[\phi] \omega_y + \cos[\phi] \sec[\theta] \omega_z \right\}, \{\omega_x, \omega_y, \omega_z\} \right]$$

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot{\phi} - \dot{\psi} \sin[\theta] \\ \frac{1}{2} (2 \cos[\phi] \dot{\theta} - \dot{\psi} \sin[\theta - \phi] + \dot{\psi} \sin[\theta + \phi]) \\ \frac{1}{2} (\cos[\theta - \phi] \dot{\psi} + \cos[\theta + \phi] \dot{\psi} - 2 \dot{\theta} \sin[\phi]) \end{pmatrix} = \begin{pmatrix} \dot{\phi} - \dot{\psi} \sin[\theta] \\ \cos[\phi] \dot{\theta} + \cos[\theta] \dot{\psi} \sin[\phi] \\ \cos[\theta] \cos[\phi] \dot{\psi} - \dot{\theta} \sin[\phi] \end{pmatrix}$$

RPY angles



$$R[\phi] = R_x[\phi] R_y[\theta] R_x[\psi]$$

$$= \begin{pmatrix} \cos[\phi] & -\sin[\phi] & 0 \\ \sin[\phi] & \cos[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\psi] & -\sin[\psi] \\ 0 & \sin[\psi] & \cos[\psi] \end{pmatrix} // \text{MatrixForm}$$

$$= \begin{pmatrix} \cos[\theta] \cos[\phi] & -\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\phi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\psi] & \cos[\theta] \cos[\psi] \end{pmatrix}$$

Euler Angle (Roll ϕ $\hat{x}_B$, Pitch θ $\hat{y}_B$, Yaw ψ $\hat{z}_B$, ZYX) $\rightarrow$ Rotation Matrix ${}^A_B R$

$${}^A_B R = R_x[\psi] \cdot R_y[\theta] \cdot R_x[\phi]$$

$$= \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$= \begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\theta] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

■ Inverse problem

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

the three RPY angles are ($\theta \in (-\pi/2, \pi/2)$)

$$\phi = \text{ArcTan}[r_{11}, r_{21}]$$

$$\theta = \text{ArcTan}[\sqrt{r_{32}^2 + r_{33}^2}, -r_{31}] = \text{ArcSin}[-r_{31}]$$

$$\psi = \text{ArcTan}[r_{33}, r_{32}]$$

or ($\theta \in (\pi/2, 3\pi/2)$)

$$\phi = \text{ArcTan}[-r_{11}, -r_{21}]$$

$$\theta = \text{ArcTan}[-\sqrt{r_{32}^2 + r_{33}^2}, -r_{31}] = \text{ArcSin}[-r_{31}]$$

$$\psi = \text{ArcTan}[-r_{33}, -r_{32}]$$

■ θ (Pitch) = $\pi/2$, $\cos[\theta] \approx 0$

$$r_{23} - r_{12} = (\sin[\theta] + 1) \sin[\phi - \psi]$$

$$r_{13} + r_{22} = (\sin[\theta] + 1) \cos[\phi - \psi]$$

$$r_{23} + r_{12} = (\sin[\theta] - 1) \sin[\phi + \psi]$$

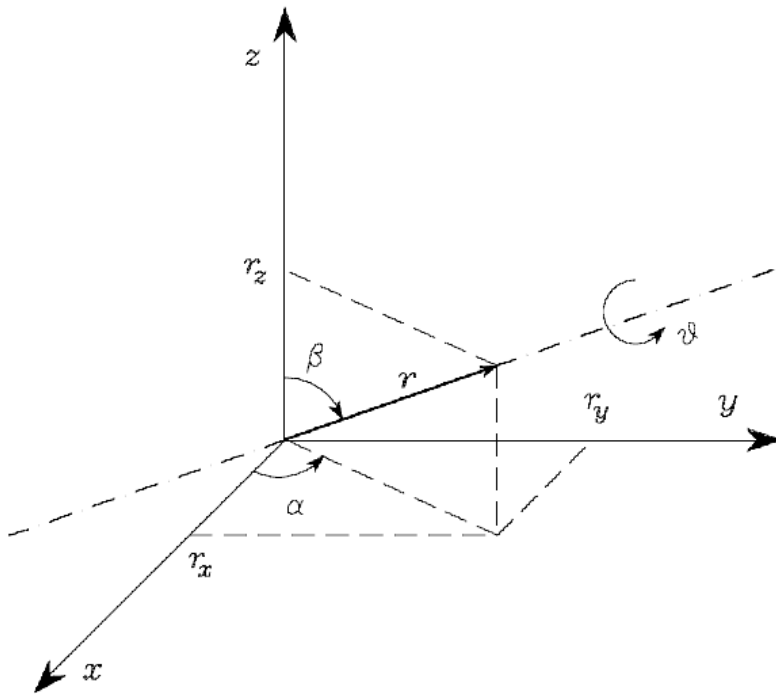
$$r_{13} - r_{22} = (\sin[\theta] + 1) \cos[\phi + \psi]$$

$$\phi = \text{ArcTan}[r_{13} - r_{22}, r_{23} + r_{12}]$$

$$\theta = \text{ArcSin}[-r_{31}]$$

$$\psi = \phi - \text{ArcTan}[r_{13} + r_{22}, r_{23} - r_{12}]$$

Angle and Axis



$$R[\theta, r] = R_x[\alpha] R_y[\beta] R_z[\theta] R_y[-\beta] R_x[-\alpha]$$

$$R[\theta, r] = R_x[\alpha] \cdot R_y[\beta] \cdot R_z[\theta] \cdot R_y[-\beta] \cdot R_x[-\alpha]$$

$$= \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & -\sin[\theta] & 0 \\ \sin[\theta] & \cos[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[-\beta] & 0 & \sin[-\beta] \\ 0 & 1 & 0 \\ -\sin[-\beta] & 0 & \cos[-\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[-\alpha] & -\sin[-\alpha] & 0 \\ \sin[-\alpha] & \cos[-\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sin[\alpha] = \frac{r_y}{\sqrt{r_x^2 + r_y^2}}$$

$$\cos[\alpha] = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\sin[\beta] = \sqrt{r_x^2 + r_y^2}$$

$$\cos[\beta] = r_z$$

$${}^A_B R = \begin{pmatrix} r_x^2 (1 - \cos[\gamma]) + \cos[\gamma] & r_x r_y (1 - \cos[\gamma]) - r_z \sin[\gamma] & r_x r_z (1 - \cos[\gamma]) + r_y \sin[\gamma] \\ r_x r_y (1 - \cos[\gamma]) + r_z \sin[\gamma] & r_y^2 (1 - \cos[\gamma]) + \cos[\gamma] & r_y r_z (1 - \cos[\gamma]) - r_x \sin[\gamma] \\ r_x r_z (1 - \cos[\gamma]) - r_y \sin[\gamma] & r_y r_z (1 - \cos[\gamma]) + r_x \sin[\gamma] & r_z^2 (1 - \cos[\gamma]) + \cos[\gamma] \end{pmatrix}$$

■ 유도

$$\sin[\alpha] = \frac{r_y}{\sqrt{r_x^2 + r_y^2}}$$

$$\cos[\alpha] = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\sin[\beta] = \sqrt{r_x^2 + r_y^2}$$

$$\cos[\beta] = r_z$$

$$\alpha = \text{ArcTan}\left[\frac{r_y}{r_x}\right];$$

$$\beta = \text{ArcTan}\left[\frac{\sqrt{r_x^2 + r_y^2}}{r_z}\right];$$

$$\begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & -\sin[\theta] & 0 \\ \sin[\theta] & \cos[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[-\beta] & 0 & \sin[-\beta] \\ 0 & 1 & 0 \\ -\sin[-\beta] & 0 & \cos[-\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[-\alpha] & -\sin[-\alpha] & 0 \\ \sin[-\alpha] & \cos[-\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} // \text{Factor} // \text{FullSimplify} // \text{MatrixForm}$$

$$\left(\begin{array}{l} \cos[\theta] - \frac{(-1 + \cos[\theta]) r_x^2}{1} - \frac{(-1 + \cos[\theta]) r_x r_y \sin[\theta] \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z^2}{1} - \frac{\sqrt{r_x^2 + r_y^2} \left((-1 + \cos[\theta]) r_x \sin[\theta] r_y \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z \right)}{r_x \sqrt{1 + \frac{r_y^2}{r_x^2}} (1)} \\ \frac{-(-1 + \cos[\theta]) r_x r_y \sin[\theta] \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z^2}{1} \cos[\theta] - \frac{(-1 + \cos[\theta]) r_y^2}{1} - \frac{\sqrt{r_x^2 + r_y^2} \left((-1 + \cos[\theta]) r_y \sin[\theta] r_x \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z \right)}{r_x \sqrt{1 + \frac{r_y^2}{r_x^2}} (1)} // \text{Factor} // \text{FullSimplify} // \text{MatrixForm} \\ \frac{\sqrt{r_x^2 + r_y^2} \left((-1 + \cos[\theta]) r_x \sin[\theta] r_y \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z \right)}{r_x \sqrt{1 + \frac{r_y^2}{r_x^2}} (1)} - \frac{\sqrt{r_x^2 + r_y^2} \left((-1 + \cos[\theta]) r_y \sin[\theta] r_x \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z \right)}{r_x \sqrt{1 + \frac{r_y^2}{r_x^2}} (1)} - \frac{\cos[\theta] (r_x^2 + r_y^2) r_z^2}{r_x^2 + r_y^2 + r_z^2} \end{array} \right)$$

$$\begin{pmatrix} \cos[\theta] - (-1 + \cos[\theta]) r_x^2 & -(-1 + \cos[\theta]) r_x r_y - \sin[\theta] \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z^2 & \frac{\sqrt{r_x^2 + r_y^2} \left(-(-1 + \cos[\theta]) r_x \sin[\theta] r_y \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} \right) r_z}{\sqrt{r_x^2 + r_y^2}} \\ -(-1 + \cos[\theta]) r_x r_y + \sin[\theta] \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z^2 & \cos[\theta] - (-1 + \cos[\theta]) r_y^2 & \frac{\sqrt{r_x^2 + r_y^2} \left(-(-1 + \cos[\theta]) r_y \sin[\theta] r_x \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} \right) r_z}{\sqrt{r_x^2 + r_y^2}} \\ \frac{\sqrt{r_x^2 + r_y^2} \left(-(-1 + \cos[\theta]) r_x \sin[\theta] r_y \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} \right) r_z}{\sqrt{r_x^2 + r_y^2}} & \frac{\sqrt{r_x^2 + r_y^2} \left(-(-1 + \cos[\theta]) r_y \sin[\theta] r_x \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} \right) r_z}{\sqrt{r_x^2 + r_y^2}} & \frac{\cos[\theta] (r_x^2 + r_y^2) + r_z^2}{r_x^2 + r_y^2 + r_z^2} \end{pmatrix} // \text{Expand} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] + r_x^2 - \cos[\theta] r_x^2 & r_x r_y - \cos[\theta] r_x r_y - \sin[\theta] \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z^2 & r_x r_z - \cos[\theta] r_x r_z + \sin[\theta] r_y \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z \\ r_x r_y - \cos[\theta] r_x r_y + \sin[\theta] \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z^2 & \cos[\theta] + r_y^2 - \cos[\theta] r_y^2 & r_y r_z - \cos[\theta] r_y r_z - \sin[\theta] r_x \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z \\ r_x r_z - \cos[\theta] r_x r_z - \sin[\theta] r_y \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z & r_y r_z - \cos[\theta] r_y r_z + \sin[\theta] r_x \sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z & \frac{\cos[\theta] r_x^2}{r_x^2 + r_y^2 + r_z^2} + \frac{\cos[\theta] r_y^2}{r_x^2 + r_y^2 + r_z^2} + \frac{r_z^2}{r_x^2 + r_y^2 + r_z^2} \end{pmatrix} // \text{FullSimplify} // \text{MatrixForm}$$

$$r_x^2 + r_y^2 + r_z^2 = 1;$$

$$\sqrt{1 + \frac{r_x^2 + r_y^2}{r_z^2}} r_z = 1;$$

$$\begin{pmatrix} r_x^2 (1 - \cos[\theta]) + \cos[\theta] & r_x r_y (1 - \cos[\theta]) - r_z \sin[\theta] & r_x r_z (1 - \cos[\theta]) + r_y \sin[\theta] \\ r_x r_y (1 - \cos[\theta]) + r_z \sin[\theta] & r_y^2 (1 - \cos[\theta]) + \cos[\theta] & r_y r_z (1 - \cos[\theta]) - r_x \sin[\theta] \\ r_x r_z (1 - \cos[\theta]) - r_y \sin[\theta] & r_y r_z (1 - \cos[\theta]) + r_x \sin[\theta] & r_z^2 (1 - \cos[\theta]) + \cos[\theta] \end{pmatrix}$$

■ Inverse problem

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

the angle and axis of rotation are ($\sin[\theta] \neq 0$)

$$\theta = \cos \left[\frac{r_{11} + r_{22} + r_{33} - 1}{2} \right]^{-1}$$

$$r = \frac{1}{2 \sin[\theta]} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

$$r_x^2 + r_y^2 + r_z^2 = 1$$

Angle θ and Axis r

$$r = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}$$

$$| \mathbf{r} | = \sqrt{r_x^2 + r_y^2 + r_z^2} = 1$$

$$\sin[\alpha] = \frac{r_y}{\sqrt{r_x^2 + r_y^2}}$$

$$\cos[\alpha] = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\sin[\beta] = \frac{\sqrt{r_x^2 + r_y^2}}{\sqrt{r_x^2 + r_y^2 + r_z^2}} = \sqrt{r_x^2 + r_y^2}$$

$$\cos[\beta] = \frac{r_z}{\sqrt{r_x^2 + r_y^2 + r_z^2}} = r_z$$

$$\alpha = \text{ArcTan}\left[\frac{r_y}{r_x}\right]$$

$$\beta = \text{ArcTan}\left[\frac{\sqrt{r_x^2 + r_y^2}}{r_z}\right]$$

$$\mathbf{R}[\theta, \mathbf{r}] = \mathbf{R}_z[\alpha] \cdot \mathbf{R}_y[\beta] \cdot \mathbf{R}_x[\theta] \cdot \mathbf{R}_y[-\beta] \cdot \mathbf{R}_z[-\alpha]$$

$$= \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & -\sin[\theta] & 0 \\ \sin[\theta] & \cos[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[-\beta] & 0 & \sin[-\beta] \\ 0 & 1 & 0 \\ -\sin[-\beta] & 0 & \cos[-\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[-\alpha] & -\sin[-\alpha] & 0 \\ \sin[-\alpha] & \cos[-\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} r_x^2 (1 - \cos[\theta]) + \cos[\theta] & r_x r_y (1 - \cos[\theta]) - r_z \sin[\theta] & r_x r_z (1 - \cos[\theta]) + r_y \sin[\theta] \\ r_x r_y (1 - \cos[\theta]) + r_z \sin[\theta] & r_y^2 (1 - \cos[\theta]) + \cos[\theta] & r_y r_z (1 - \cos[\theta]) - r_x \sin[\theta] \\ r_x r_z (1 - \cos[\theta]) - r_y \sin[\theta] & r_y r_z (1 - \cos[\theta]) + r_x \sin[\theta] & r_z^2 (1 - \cos[\theta]) + \cos[\theta] \end{pmatrix}$$

Rotation Matrix → Angle θ and Axis $\mathbf{r}$

$$\mathbf{R}[\theta, \mathbf{r}] = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} r_x^2 (1 - \cos[\theta]) + \cos[\theta] & r_x r_y (1 - \cos[\theta]) - r_z \sin[\theta] & r_x r_z (1 - \cos[\theta]) + r_y \sin[\theta] \\ r_x r_y (1 - \cos[\theta]) + r_z \sin[\theta] & r_y^2 (1 - \cos[\theta]) + \cos[\theta] & r_y r_z (1 - \cos[\theta]) - r_x \sin[\theta] \\ r_x r_z (1 - \cos[\theta]) - r_y \sin[\theta] & r_y r_z (1 - \cos[\theta]) + r_x \sin[\theta] & r_z^2 (1 - \cos[\theta]) + \cos[\theta] \end{pmatrix}$$

$$\mathbf{R}[-\theta, -\mathbf{r}] = \mathbf{R}[\theta, \mathbf{r}]$$

$$\theta = \text{ArcCos}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right]$$

$$\mathbf{r} = \frac{1}{2 \sin[\theta]} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

▪ 유도

▪ 1

$$r_{11} + r_{22} + r_{33}$$

$$= r_x^2 (1 - \cos[\theta]) + \cos[\theta] + r_y^2 (1 - \cos[\theta]) + \cos[\theta] + r_s^2 (1 - \cos[\theta]) + \cos[\theta]$$

$$= (r_x^2 + r_y^2 + r_s^2) (1 - \cos[\theta]) + 3 \cos[\theta]$$

$$= (1 - \cos[\theta]) + 3 \cos[\theta]$$

$$= 1 + 2 \cos[\theta]$$

$$\cos[\theta] = \frac{r_{11} + r_{22} + r_{33} - 1}{2}$$

$$\theta = \text{ArcCos}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right]$$

■ 2

$$r_{32} - r_{23}$$

$$= (r_y r_s (1 - \cos[\theta]) + r_x \sin[\theta]) - (r_y r_s (1 - \cos[\theta]) - r_x \sin[\theta])$$

$$= 2 r_x \sin[\theta]$$

$$r_x = \frac{r_{32} - r_{23}}{2 \sin[\theta]}$$

■ 3

$$r_{13} - r_{31}$$

$$= (r_x r_s (1 - \cos[\theta]) + r_y \sin[\theta]) - (r_x r_s (1 - \cos[\theta]) - r_y \sin[\theta])$$

$$= 2 r_y \sin[\theta]$$

$$r_y = \frac{r_{13} - r_{31}}{2 \sin[\theta]}$$

■ 4

$$r_{21} - r_{12}$$

$$= (r_x r_y (1 - \cos[\theta]) + r_s \sin[\theta]) - (r_x r_y (1 - \cos[\theta]) - r_s \sin[\theta])$$

$$= 2 r_s \sin[\theta]$$

$$r_s = \frac{r_{21} - r_{12}}{2 \sin[\theta]}$$

■ 5

$$\theta = \text{ArcCos}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right]$$

$$\hat{\mathbf{r}} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \frac{1}{2 \sin[\theta]} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

$$r_x^2 + r_y^2 + r_z^2 = 1$$

Quaternion → Rotation Matrix

- Euler axis

$$\hat{\mathbf{u}} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$

- Angle

$$\theta$$

- Rotation Matrix

$$\mathbf{R}[\theta_] := \begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_y u_x (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_z u_x (1 - \cos[\theta]) - u_y \sin[\theta] & u_z u_y (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix}$$

- Concisely

$$\mathbf{R} = \mathbf{I} \cos[\theta] + \sin[\theta] [\mathbf{u}]_{\times} + (1 - \cos[\theta]) \mathbf{u} \otimes \mathbf{u}$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[\mathbf{u}]_{\times} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}$$

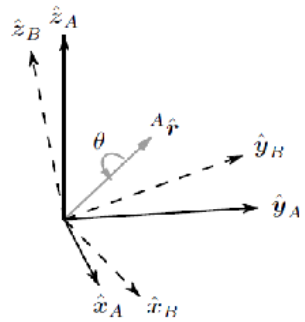
$$\mathbf{u} \otimes \mathbf{u} = \begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{pmatrix}$$

Quaternion Representation

- A quaternion is a four-dimensional complex number

- An arbitrary orientation of frame B relative to frame A

$${}^A_B \hat{\mathbf{q}}$$



- A rotation of angle θ around an axis ${}^A\hat{r}$ defined in frame A

$${}^A\hat{r} = (r_x \quad r_y \quad r_z)$$

- normalized four-dimensional vector

$${}^B\hat{q} = (q_1 \quad q_2 \quad q_3 \quad q_4) = \left(\cos\left[\frac{\theta}{2}\right] \quad -r_x \sin\left[\frac{\theta}{2}\right] \quad -r_y \sin\left[\frac{\theta}{2}\right] \quad -r_z \sin\left[\frac{\theta}{2}\right] \right)$$

- 4parameter representation $\rho=\{\eta,\epsilon\}$

$$\eta = \cos\left[\frac{\theta}{2}\right]$$

$$\epsilon = \sin\left[\frac{\theta}{2}\right] {}^A\hat{r}$$

$$(\theta, {}^A\hat{r}) \text{ and } (-\theta, -{}^A\hat{r}) \text{ give the same quaternion}$$

- quaternion extracted from $R^{-1}=R^T$

$$q^{-1} = \{\eta, -\epsilon\}$$

- the quaternion parametrization obeys

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

$$\eta^2 + \epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2 = 1$$

- hypercomplex numbers $\{i,j,k\}$

$$a + b i + c j + d k$$

$$i^2 = j^2 = k^2 = -1$$

$$i j = -j i = k$$

$$j k = -k j = i$$

$$\mathbf{k} \mathbf{i} = -\mathbf{i} \mathbf{k} = \mathbf{j}$$

- 2x2 Complex Matrices, The quaternion $a + b \mathbf{i} + c \mathbf{j} + d \mathbf{k}$

$$\begin{pmatrix} a + b \mathbf{i} & c + d \mathbf{i} \\ -c + d \mathbf{i} & a - b \mathbf{i} \end{pmatrix}$$

- A quaternion conjugate

$$\hat{\mathbf{A}}_{\mathbf{B}} \hat{\mathbf{Q}}^* = \mathbf{B}_{\mathbf{A}} \hat{\mathbf{Q}} = (\mathbf{q}_1 \quad -\mathbf{q}_2 \quad -\mathbf{q}_3 \quad -\mathbf{q}_4)$$

- A quaternion product $\otimes$

$$\hat{\mathbf{A}}_{\mathbf{C}} \hat{\mathbf{Q}} = \mathbf{B}_{\mathbf{C}} \hat{\mathbf{Q}} \otimes \mathbf{A}_{\mathbf{B}} \hat{\mathbf{Q}}$$

$$\varrho_1 = \{\eta_1, \epsilon_1\}, \varrho_2 = \{\eta_2, \epsilon_2\}$$

$$\varrho_1 * \varrho_2 = \{\eta_1 \eta_2 - \epsilon_1^\top \epsilon_2, \eta_1 \epsilon_2 + \eta_2 \epsilon_1 + \epsilon_1 \times \epsilon_2\}$$

- Hamilton rule

$$\mathbf{a} \otimes \mathbf{b} = (\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4) \otimes (\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4)$$

$$= (\mathbf{a}_1 + \mathbf{a}_2 \mathbf{i} + \mathbf{a}_3 \mathbf{j} + \mathbf{a}_4 \mathbf{k}) (\mathbf{b}_1 + \mathbf{b}_2 \mathbf{i} + \mathbf{b}_3 \mathbf{j} + \mathbf{b}_4 \mathbf{k})$$

$$= \mathbf{a}_1 \mathbf{b}_1 + \mathbf{i} \mathbf{a}_2 \mathbf{b}_1 + \mathbf{j} \mathbf{a}_3 \mathbf{b}_1 + \mathbf{k} \mathbf{a}_4 \mathbf{b}_1 + \mathbf{i} \mathbf{a}_1 \mathbf{b}_2 + \mathbf{i}^2 \mathbf{a}_2 \mathbf{b}_2 + \mathbf{i} \mathbf{j} \mathbf{a}_3 \mathbf{b}_2 + \mathbf{i} \mathbf{k} \mathbf{a}_4 \mathbf{b}_2 + \mathbf{j} \mathbf{a}_1 \mathbf{b}_3 + \mathbf{i} \mathbf{j} \mathbf{a}_2 \mathbf{b}_3 + \mathbf{j}^2 \mathbf{a}_3 \mathbf{b}_3 + \mathbf{j} \mathbf{k} \mathbf{a}_4 \mathbf{b}_3 + \mathbf{k} \mathbf{a}_1 \mathbf{b}_4 + \mathbf{i} \mathbf{k} \mathbf{a}_2 \mathbf{b}_4 + \mathbf{j} \mathbf{k} \mathbf{a}_3 \mathbf{b}_4 + \mathbf{k}^2 \mathbf{a}_4 \mathbf{b}_4$$

$$= \mathbf{a}_1 \mathbf{b}_1 + \mathbf{i}^2 \mathbf{a}_2 \mathbf{b}_2 + \mathbf{j}^2 \mathbf{a}_3 \mathbf{b}_3 + \mathbf{k}^2 \mathbf{a}_4 \mathbf{b}_4$$

$$+ \mathbf{i} \mathbf{a}_2 \mathbf{b}_1 + \mathbf{i} \mathbf{a}_1 \mathbf{b}_2 + \mathbf{j} \mathbf{k} \mathbf{a}_4 \mathbf{b}_3 + \mathbf{j} \mathbf{k} \mathbf{a}_3 \mathbf{b}_4$$

$$+ \mathbf{j} \mathbf{a}_3 \mathbf{b}_1 + \mathbf{i} \mathbf{k} \mathbf{a}_4 \mathbf{b}_2 + \mathbf{j} \mathbf{a}_1 \mathbf{b}_3 + \mathbf{i} \mathbf{k} \mathbf{a}_2 \mathbf{b}_4$$

$$+ \mathbf{k} \mathbf{a}_4 \mathbf{b}_1 + \mathbf{i} \mathbf{j} \mathbf{a}_3 \mathbf{b}_2 + \mathbf{i} \mathbf{j} \mathbf{a}_2 \mathbf{b}_3 + \mathbf{k} \mathbf{a}_1 \mathbf{b}_4$$

$$= \mathbf{a}_1 \mathbf{b}_1 - \mathbf{a}_2 \mathbf{b}_2 - \mathbf{a}_3 \mathbf{b}_3 - \mathbf{a}_4 \mathbf{b}_4$$

$$+ \mathbf{i} \mathbf{a}_2 \mathbf{b}_1 + \mathbf{i} \mathbf{a}_1 \mathbf{b}_2 - \mathbf{i} \mathbf{a}_4 \mathbf{b}_3 + \mathbf{i} \mathbf{a}_3 \mathbf{b}_4$$

$$+ \mathbf{j} \mathbf{a}_3 \mathbf{b}_1 + \mathbf{j} \mathbf{a}_4 \mathbf{b}_2 + \mathbf{j} \mathbf{a}_1 \mathbf{b}_3 - \mathbf{j} \mathbf{a}_2 \mathbf{b}_4$$

$$+ \mathbf{k} \mathbf{a}_4 \mathbf{b}_1 - \mathbf{k} \mathbf{a}_3 \mathbf{b}_2 + \mathbf{k} \mathbf{a}_2 \mathbf{b}_3 + \mathbf{k} \mathbf{a}_1 \mathbf{b}_4$$

$$= \mathbf{a}_1 \mathbf{b}_1 - \mathbf{a}_2 \mathbf{b}_2 - \mathbf{a}_3 \mathbf{b}_3 - \mathbf{a}_4 \mathbf{b}_4$$

$$+ \mathbf{i} (\mathbf{a}_1 \mathbf{b}_2 + \mathbf{a}_2 \mathbf{b}_1 + \mathbf{a}_3 \mathbf{b}_4 - \mathbf{a}_4 \mathbf{b}_3)$$

$$+ \mathbf{j} (\mathbf{a}_1 \mathbf{b}_3 - \mathbf{a}_2 \mathbf{b}_4 + \mathbf{a}_3 \mathbf{b}_1 + \mathbf{a}_4 \mathbf{b}_2)$$

$$+ \mathbf{k} (\mathbf{a}_1 \mathbf{b}_4 + \mathbf{a}_2 \mathbf{b}_3 - \mathbf{a}_3 \mathbf{b}_2 + \mathbf{a}_4 \mathbf{b}_1)$$

$$= \begin{pmatrix} a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4 \\ a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3 \\ a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2 \\ a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1 \end{pmatrix}^T$$

$$= (a_1 \ a_2 \ a_3 \ a_4) \cdot \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ -b_2 & b_1 & -b_4 & b_3 \\ -b_3 & b_4 & b_1 & -b_2 \\ -b_4 & -b_3 & b_2 & b_1 \end{pmatrix}$$

- Euler axis

$$\hat{e} = \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}$$

$$e = \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}$$

- Angle

$$\theta$$

- Vector's Elements

$$q_1 = \cos\left[\frac{\theta}{2}\right]$$

$$q_2 = e_x \sin\left[\frac{\theta}{2}\right]$$

$$q_3 = e_y \sin\left[\frac{\theta}{2}\right]$$

$$q_4 = e_z \sin\left[\frac{\theta}{2}\right]$$

- Inverse problem

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

the quaternion is ($\eta \geq 0$)

$$\eta = \frac{1}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1}$$

$$e = \frac{1}{2} \begin{pmatrix} \text{Sign}[r_{32} - r_{23}] \sqrt{r_{11} - r_{22} - r_{33} + 1} \\ \text{Sign}[r_{13} - r_{31}] \sqrt{r_{22} - r_{33} - r_{11} + 1} \\ \text{Sign}[r_{21} - r_{12}] \sqrt{r_{33} - r_{11} - r_{22} + 1} \end{pmatrix}$$

- A three dimensional vector - Same vector from frame A to frame B

$$\mathbf{a} \otimes \mathbf{b} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ -b_2 & b_1 & -b_4 & b_3 \\ -b_3 & b_4 & b_1 & -b_2 \\ -b_4 & -b_3 & b_2 & b_1 \end{pmatrix}$$

$$\mathbb{B}_U = \mathbb{A}_B \hat{Q} \otimes \mathbb{A}_U \otimes \mathbb{A}_B \hat{Q}^*$$

$$= \begin{pmatrix} 0 & \mathbb{B}_{U_X} & \mathbb{B}_{U_Y} & \mathbb{B}_{U_S} \end{pmatrix} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & \mathbb{A}_{U_X} & \mathbb{A}_{U_Y} & \mathbb{A}_{U_S} \end{pmatrix} \otimes \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \end{pmatrix}$$

$$= \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & \mathbb{A}_{U_X} & \mathbb{A}_{U_Y} & \mathbb{A}_{U_S} \end{pmatrix} \otimes \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \end{pmatrix}$$

$$= \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & v_X & v_Y & v_S \end{pmatrix} \otimes \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \end{pmatrix}$$

$$= \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} \cdot \begin{pmatrix} 0 & v_X & v_Y & v_S \\ -v_X & 0 & -v_S & v_Y \\ -v_Y & v_S & 0 & -v_X \\ -v_S & -v_Y & v_X & 0 \end{pmatrix} \cdot \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & q_4 & -q_3 \\ q_3 & -q_4 & q_1 & q_2 \\ q_4 & q_3 & -q_2 & q_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ q_1^2 v_X + q_2^2 v_X - q_3^2 v_X - q_4^2 v_X + 2 q_2 q_3 v_Y - 2 q_1 q_4 v_Y + 2 q_1 q_3 v_S + 2 q_2 q_4 v_S \\ 2 q_2 q_3 v_X + 2 q_1 q_4 v_X + q_1^2 v_Y - q_2^2 v_Y + q_3^2 v_Y - q_4^2 v_Y - 2 q_1 q_2 v_S + 2 q_3 q_4 v_S \\ -2 q_1 q_3 v_X + 2 q_2 q_4 v_X + 2 q_1 q_2 v_Y + 2 q_3 q_4 v_Y + q_1^2 v_S - q_2^2 v_S - q_3^2 v_S + q_4^2 v_S \end{pmatrix}_T$$

$$= \begin{pmatrix} 0 \\ (q_1^2 + q_2^2 - q_3^2 - q_4^2) v_X + 2 (q_2 q_3 - q_1 q_4) v_Y + 2 (q_1 q_3 + q_2 q_4) v_S \\ 2 (q_2 q_3 + q_1 q_4) v_X + (q_1^2 - q_2^2 + q_3^2 - q_4^2) v_Y + 2 (-q_1 q_2 + q_3 q_4) v_S \\ 2 (-q_1 q_3 + q_2 q_4) v_X + 2 (q_1 q_2 + q_3 q_4) v_Y + (q_1^2 - q_2^2 - q_3^2 + q_4^2) v_S \end{pmatrix}_T$$

■ 0

$$\mathbb{B}_U = \mathbb{A}_B \hat{Q} \otimes \mathbb{A}_U \otimes \mathbb{A}_B \hat{Q}^*$$

$$\mathbb{A}_B \mathbf{R} = \begin{pmatrix} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 - q_1 q_4) & 2 (q_2 q_4 - q_1 q_3) \\ 2 (q_2 q_3 - q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 + q_1 q_2) \\ 2 (q_2 q_4 + q_1 q_3) & 2 (q_3 q_4 - q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{pmatrix}$$

$$\mathbb{A}_A \mathbf{R} = \begin{pmatrix} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 - q_1 q_4) & 2 (q_2 q_4 + q_1 q_3) \\ 2 (q_2 q_3 + q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 - q_1 q_2) \\ 2 (q_2 q_4 - q_1 q_3) & 2 (q_3 q_4 + q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{pmatrix}$$

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

$$\mathbb{B}_U = \mathbb{A}_A \mathbf{R} \cdot \mathbb{A}_U$$

$$= \begin{pmatrix} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 - q_1 q_4) & 2 (q_2 q_4 + q_1 q_3) \\ 2 (q_2 q_3 + q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 - q_1 q_2) \\ 2 (q_2 q_4 - q_1 q_3) & 2 (q_3 q_4 + q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{pmatrix} \cdot \begin{pmatrix} v_X \\ v_Y \\ v_S \end{pmatrix}$$

$$= \begin{pmatrix} (-1 + 2 q_1^2 + 2 q_2^2) v_X + 2 (q_2 q_3 - q_1 q_4) v_Y + 2 (q_1 q_3 + q_2 q_4) v_S \\ 2 (q_2 q_3 + q_1 q_4) v_X + (-1 + 2 q_1^2 + 2 q_3^2) v_Y + 2 (-q_1 q_2 + q_3 q_4) v_S \\ 2 (-q_1 q_3 + q_2 q_4) v_X + 2 (q_1 q_2 + q_3 q_4) v_Y + (-1 + 2 q_1^2 + 2 q_4^2) v_S \end{pmatrix}$$

$$= \begin{pmatrix} (- (q_1^2 + q_2^2 + q_3^2 + q_4^2) + 2 q_1^2 + 2 q_2^2) v_x + 2 (q_2 q_3 - q_1 q_4) v_y + 2 (q_1 q_3 + q_2 q_4) v_z \\ 2 (q_2 q_3 + q_1 q_4) v_x + (- (q_1^2 + q_2^2 + q_3^2 + q_4^2) + 2 q_1^2 + 2 q_3^2) v_y + 2 (- q_1 q_2 + q_3 q_4) v_z \\ 2 (- q_1 q_3 + q_2 q_4) v_x + 2 (q_1 q_2 + q_3 q_4) v_y + (- (q_1^2 + q_2^2 + q_3^2 + q_4^2) + 2 q_1^2 + 2 q_4^2) v_z \end{pmatrix}$$

$$= \begin{pmatrix} (q_1^2 + q_2^2 - q_3^2 - q_4^2) v_x + 2 (q_2 q_3 - q_1 q_4) v_y + 2 (q_1 q_3 + q_2 q_4) v_z \\ 2 (q_2 q_3 + q_1 q_4) v_x + (q_1^2 - q_2^2 + q_3^2 - q_4^2) v_y + 2 (- q_1 q_2 + q_3 q_4) v_z \\ 2 (- q_1 q_3 + q_2 q_4) v_x + 2 (q_1 q_2 + q_3 q_4) v_y + (q_1^2 - q_2^2 - q_3^2 + q_4^2) v_z \end{pmatrix}$$

Rodrigues formula

■ Vector

$$\mathbf{a} \times \mathbf{b} \neq \mathbf{a} \times \mathbf{b}$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \times \mathbf{b} - \mathbf{a} \cdot \mathbf{b}$$

$$\mathbf{x} \times \mathbf{y} \times \mathbf{z} = (\mathbf{x} \cdot \mathbf{z}) \mathbf{y} - (\mathbf{x} \cdot \mathbf{y}) \mathbf{z}$$

$$\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c} = (\mathbf{a} \times \mathbf{b} - \mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$$

$$= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$$

$$= \{ (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \} - (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$$

$$= \{ (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \} - (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$$

$$= (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} - 2 (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$$

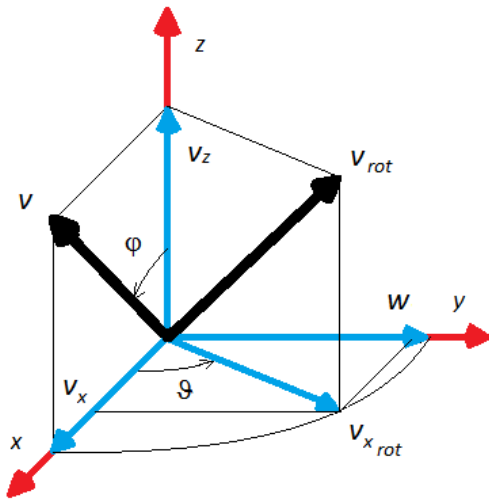
$$\mathbf{u} \cdot \mathbf{v} \cdot \mathbf{u} = (\mathbf{u} \cdot \mathbf{u}) \mathbf{v} - (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} - 2 (\mathbf{u} \cdot \mathbf{v}) \mathbf{u}$$

$$= (\mathbf{u} \cdot \mathbf{u}) \mathbf{v} - 2 (\mathbf{u} \cdot \mathbf{v}) \mathbf{u}$$

$$(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = (\mathbf{z} \times \mathbf{x}) \cdot \mathbf{y} = (\mathbf{y} \times \mathbf{z}) \cdot \mathbf{x}$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{u}) \cdot \mathbf{v} = (\mathbf{v} \times \mathbf{u}) \cdot \mathbf{u} = 0$$

■ 1



$$v_{rot} = v \cos[\theta] + (k \times v) \sin[\theta] + k (k \cdot v) (1 - \cos[\theta])$$

Given a rotation axis represented by a unit vector k and a vector v to rotate about k by the angle θ

- Vector Projection of v on k

$$v_{parallel} = (k \cdot v) k$$

- Vector rejection of v from k

$$v_{orthogonal} = v - v_{parallel} = v - (k \cdot v) k$$

$$w = k \times v = k \times (v_{orthogonal} + v_{parallel}) = k \times v_{orthogonal} + k \times v_{parallel} = k \times v_{orthogonal}$$

$$v_{orthogonal_{rot}} = v_{orthogonal} \cos[\theta] + w \sin[\theta]$$

$$= (v - (k \cdot v) k) \cos[\theta] + (k \times v) \sin[\theta]$$

$$v_{rot} = v_{orthogonal_{rot}} + v_{parallel_{rot}}$$

$$= v_{orthogonal_{rot}} + v_{parallel}$$

$$= (v - (k \cdot v) k) \cos[\theta] + (k \times v) \sin[\theta] + (k \cdot v) k$$

$$= v \cos[\theta] + (k \times v) \sin[\theta] + k (k \cdot v) (1 - \cos[\theta])$$

$$K = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix}$$

$$K v = k \times v$$

$$\mathbf{v}_{\text{rot}} = \mathbf{v} \cos[\theta] + (\mathbf{k} \times \mathbf{v}) \sin[\theta] + \mathbf{k} (\mathbf{k} \cdot \mathbf{v}) (1 - \cos[\theta])$$

$$= \mathbf{v} \cos[\theta] + K \mathbf{v} \sin[\theta] + \mathbf{k} (\mathbf{k} \cdot \mathbf{v}) (1 - \cos[\theta])$$

$$= \mathbf{v} - \mathbf{v} + \mathbf{v} \cos[\theta] + K \mathbf{v} \sin[\theta] + \mathbf{k} (\mathbf{k} \cdot \mathbf{v}) (1 - \cos[\theta])$$

$$= \mathbf{v} - \mathbf{v} (1 - \cos[\theta]) + K \mathbf{v} \sin[\theta] + \mathbf{k} (\mathbf{k} \cdot \mathbf{v}) (1 - \cos[\theta])$$

$$= \mathbf{v} + K \mathbf{v} \sin[\theta] + (\mathbf{k} (\mathbf{k} \cdot \mathbf{v}) - \mathbf{v}) (1 - \cos[\theta])$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} (\mathbf{a} \cdot \mathbf{b})$$

$$(\mathbf{k} (\mathbf{k} \cdot \mathbf{v}) - \mathbf{v}) = (\mathbf{k} (\mathbf{k} \cdot \mathbf{v}) - \mathbf{v} (\mathbf{k} \cdot \mathbf{k})) = \mathbf{k} \times (\mathbf{k} \times \mathbf{v}) = K^2 \mathbf{v}$$

$$= \mathbf{v} + \sin[\theta] K \mathbf{v} + (1 - \cos[\theta]) K^2 \mathbf{v}$$

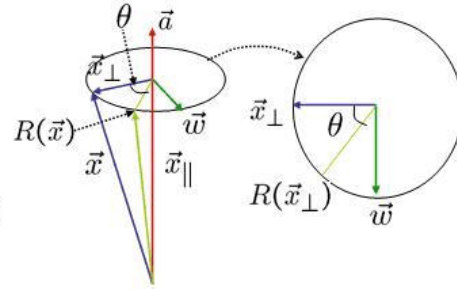
$$= R \mathbf{v}$$

$$R = I + \sin[\theta] K + (1 - \cos[\theta]) K^2$$

$$R = \exp[\theta K]$$

$$R[\theta] R[\phi] = R[\theta + \phi], R[0] = I$$

$$\begin{aligned} \vec{w} &= \vec{a} \times \vec{x}_{\perp} \\ &= \vec{a} \times (\vec{x} - \vec{x}_{\parallel}) \\ &= (\vec{a} \times \vec{x}) - (\vec{a} \times \vec{x}_{\parallel}) \\ &= \vec{a} \times \vec{x} \end{aligned}$$



$$R(\vec{x}_{\perp}) = \cos \theta \vec{x}_{\perp} + \sin \theta \vec{w}$$

$$\vec{x}_{\parallel} = (\vec{a} \cdot \vec{x}) \vec{a}$$

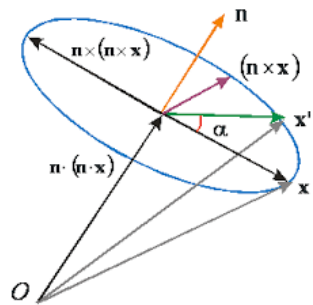
$$\vec{x}_{\perp} = \vec{x} - \vec{x}_{\parallel} = \vec{x} - (\vec{a} \cdot \vec{x}) \vec{a}$$

$$R(\vec{x}) = R(\vec{x}_{\parallel}) + R(\vec{x}_{\perp})$$

$$= R(\vec{x}_{\parallel}) + \cos \theta \vec{x}_{\perp} + \sin \theta \vec{w}$$

$$= (\vec{a} \cdot \vec{x}) \vec{a} + \cos \theta (\vec{x} - (\vec{a} \cdot \vec{x}) \vec{a}) + \sin \theta \vec{w}$$

$$= \cos \theta \vec{x} + (1 - \cos \theta) (\vec{a} \cdot \vec{x}) \vec{a} + \sin \theta (\vec{a} \times \vec{x})$$



$$\mathbf{x} = \mathbf{n} \cdot (\mathbf{n} \cdot \mathbf{x}) - \mathbf{n} \times (\mathbf{n} \times \mathbf{x})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} (\mathbf{a} \cdot \mathbf{b})$$

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{x}) = \mathbf{n} (\mathbf{n} \cdot \mathbf{x}) - \mathbf{x} (\mathbf{n} \cdot \mathbf{n}) = \mathbf{n} (\mathbf{n} \cdot \mathbf{x}) - \mathbf{x}$$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$\mathbf{x}' = \mathbf{n} \cdot (\mathbf{n} \cdot \mathbf{x}) + \sin[\alpha] (\mathbf{n} \times \mathbf{x}) - \cos[\alpha] \mathbf{n} \times (\mathbf{n} \times \mathbf{x})$$

$$\mathbf{x}' = \mathbf{x} + \sin[\alpha] \mathbf{N} \mathbf{x} + (1 - \cos[\alpha]) \mathbf{N}^2 \mathbf{x}$$

$$\mathbf{x}' = \mathbf{n} \cdot (\mathbf{n} \cdot \mathbf{x}) + \sin[\alpha] (\mathbf{n} \times \mathbf{x}) - \cos[\alpha] (\mathbf{n} \times (\mathbf{n} \times \mathbf{x}))$$

$$= \mathbf{n} \cdot (\mathbf{n} \cdot \mathbf{x}) + \sin[\alpha] (\mathbf{n} \times \mathbf{x}) - \cos[\alpha] ((\mathbf{n} \cdot \mathbf{x}) \mathbf{n} - \mathbf{x})$$

$$= (1 - \cos[\alpha]) \mathbf{n} (\mathbf{n} \cdot \mathbf{x}) + \sin[\alpha] (\mathbf{n} \times \mathbf{x}) + \mathbf{x} \cos[\alpha]$$

$$= \mathbf{x} \cos[\alpha] + (\mathbf{n} \times \mathbf{x}) \sin[\alpha] + \mathbf{n} (\mathbf{n} \cdot \mathbf{x}) (1 - \cos[\alpha])$$

$$\cos[2\theta] = 2\cos[\theta]^2 - 1$$

$$\cos[\theta] = 2\cos\left[\frac{\theta}{2}\right]^2 - 1 = \cos\left[\frac{\theta}{2}\right]^2 - \sin\left[\frac{\theta}{2}\right]^2 = 1 - 2\sin\left[\frac{\theta}{2}\right]^2$$

$$\sin[2\theta] = 2\cos[\theta] \sin[\theta]$$

$$\sin[\theta] = 2\cos\left[\frac{\theta}{2}\right] \sin\left[\frac{\theta}{2}\right]$$

$$\mathbf{a} \mathbf{b} = \mathbf{a} \times \mathbf{b} - \mathbf{a} \cdot \mathbf{b}$$

$$\mathbf{a} \mathbf{b} \mathbf{c} = (\mathbf{a} \times \mathbf{b} - \mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

$$= (\mathbf{a} \times \mathbf{b}) \mathbf{c} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

$$= \{ (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

$$\mathbf{x} \times \mathbf{y} \times \mathbf{z} = (\mathbf{x} \cdot \mathbf{z}) \mathbf{y} - (\mathbf{x} \cdot \mathbf{y}) \mathbf{z}$$

$$= \{ (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{u}) \cdot \mathbf{v} = (\mathbf{v} \times \mathbf{u}) \cdot \mathbf{u} = 0$$

$$= (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} - 2 (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

$$\begin{aligned}\mathbf{u} \nabla \mathbf{u} &= (\mathbf{u} \cdot \mathbf{u}) \nabla - (\mathbf{u} \times \nabla) \cdot \mathbf{u} - 2 (\mathbf{u} \cdot \nabla) \mathbf{u} \\ &= (\mathbf{u} \cdot \mathbf{u}) \nabla - 2 (\mathbf{u} \cdot \nabla) \mathbf{u}\end{aligned}$$

$$\mathbf{x}' = \mathbf{n} \cdot (\mathbf{n} \cdot \mathbf{x}) + \text{Sin}[\alpha] (\mathbf{n} \times \mathbf{x}) - \text{Cos}[\alpha] (\mathbf{n} \times (\mathbf{n} \times \mathbf{x}))$$

$$= \mathbf{x} \text{Cos}[\alpha] + (\mathbf{n} \times \mathbf{x}) \text{Sin}[\alpha] + \mathbf{n} (\mathbf{n} \cdot \mathbf{x}) (1 - \text{Cos}[\alpha])$$

$$= \mathbf{x} \left(\text{Cos}\left[\frac{\alpha}{2}\right]^2 - \text{Sin}\left[\frac{\alpha}{2}\right]^2 \right) + (\mathbf{n} \times \mathbf{x}) \left(2 \text{Cos}\left[\frac{\alpha}{2}\right] \text{Sin}\left[\frac{\alpha}{2}\right] \right) + \mathbf{n} (\mathbf{n} \cdot \mathbf{x}) \left(2 \text{Sin}\left[\frac{\alpha}{2}\right]^2 \right)$$

$$= \mathbf{x} \text{Cos}\left[\frac{\alpha}{2}\right]^2 + 2 (\mathbf{n} \times \mathbf{x}) \text{Cos}\left[\frac{\alpha}{2}\right] \text{Sin}\left[\frac{\alpha}{2}\right] - (\mathbf{x} (\mathbf{n} \cdot \mathbf{n}) - 2 \mathbf{n} (\mathbf{n} \cdot \mathbf{x})) \text{Sin}\left[\frac{\alpha}{2}\right]^2$$

$$= \mathbf{x} \text{Cos}\left[\frac{\alpha}{2}\right]^2 + (\mathbf{n} \times \mathbf{x} + \mathbf{n} \times \mathbf{x}) \text{Cos}\left[\frac{\alpha}{2}\right] \text{Sin}\left[\frac{\alpha}{2}\right] - ((\mathbf{n} \cdot \mathbf{n}) \mathbf{x} - (\mathbf{n} \times \mathbf{x}) \cdot \mathbf{n} - 2 (\mathbf{n} \cdot \mathbf{x}) \mathbf{n}) \text{Sin}\left[\frac{\alpha}{2}\right]^2$$

$$= \mathbf{x} \text{Cos}\left[\frac{\alpha}{2}\right]^2 + (\mathbf{n} \times \mathbf{x} - \mathbf{x} \times \mathbf{n}) \text{Cos}\left[\frac{\alpha}{2}\right] \text{Sin}\left[\frac{\alpha}{2}\right] - (((\mathbf{n} \cdot \mathbf{n}) \mathbf{x} - (\mathbf{n} \cdot \mathbf{x}) \mathbf{n} - (\mathbf{n} \times \mathbf{x}) \cdot \mathbf{n}) - (\mathbf{n} \cdot \mathbf{x}) \mathbf{n}) \text{Sin}\left[\frac{\alpha}{2}\right]^2$$

$$= \mathbf{x} \text{Cos}\left[\frac{\alpha}{2}\right]^2 + (\mathbf{n} \times \mathbf{x} - \mathbf{x} \times \mathbf{n} + \mathbf{x} \cdot \mathbf{n} - \mathbf{n} \cdot \mathbf{x}) \text{Cos}\left[\frac{\alpha}{2}\right] \text{Sin}\left[\frac{\alpha}{2}\right] - (((\mathbf{n} \times \mathbf{x}) \times \mathbf{n} - (\mathbf{n} \times \mathbf{x}) \cdot \mathbf{n}) - (\mathbf{n} \cdot \mathbf{x}) \mathbf{n}) \text{Sin}\left[\frac{\alpha}{2}\right]^2$$

$$= \mathbf{x} \text{Cos}\left[\frac{\alpha}{2}\right]^2 + (\mathbf{n} \times \mathbf{x} - \mathbf{n} \cdot \mathbf{x} - (\mathbf{x} \times \mathbf{n} - \mathbf{x} \cdot \mathbf{n})) \text{Cos}\left[\frac{\alpha}{2}\right] \text{Sin}\left[\frac{\alpha}{2}\right] - ((\mathbf{n} \times \mathbf{x}) \mathbf{n} - (\mathbf{n} \cdot \mathbf{x}) \mathbf{n}) \text{Sin}\left[\frac{\alpha}{2}\right]^2$$

$$= \mathbf{x} \text{Cos}\left[\frac{\alpha}{2}\right]^2 + (\mathbf{n} \mathbf{x} - \mathbf{x} \mathbf{n}) \text{Cos}\left[\frac{\alpha}{2}\right] \text{Sin}\left[\frac{\alpha}{2}\right] - (\mathbf{n} \times \mathbf{x} - \mathbf{n} \cdot \mathbf{x}) \mathbf{n} \text{Sin}\left[\frac{\alpha}{2}\right]^2$$

$$= \mathbf{x} \text{Cos}\left[\frac{\alpha}{2}\right]^2 + \mathbf{n} \mathbf{x} \text{Cos}\left[\frac{\alpha}{2}\right] \text{Sin}\left[\frac{\alpha}{2}\right] - \mathbf{x} \mathbf{n} \text{Cos}\left[\frac{\alpha}{2}\right] \text{Sin}\left[\frac{\alpha}{2}\right] - \mathbf{n} \mathbf{x} \mathbf{n} \text{Sin}\left[\frac{\alpha}{2}\right]^2$$

$$= \left(\text{Cos}\left[\frac{\alpha}{2}\right] + \mathbf{n} \text{Sin}\left[\frac{\alpha}{2}\right] \right) \mathbf{x} \left(\text{Cos}\left[\frac{\alpha}{2}\right] - \mathbf{n} \text{Sin}\left[\frac{\alpha}{2}\right] \right)$$

$$= \mathbf{q} \mathbf{x} \mathbf{q}^*$$

$$\mathbf{q} = \text{Cos}\left[\frac{\alpha}{2}\right] + \mathbf{n} \text{Sin}\left[\frac{\alpha}{2}\right]$$

Axis/Angle $\rightarrow$ Rotation matrix $\rightarrow$ Orientation $\overset{\mathbf{A}}{\underset{\mathbf{B}}{\widehat{\mathbf{Q}}}}$

- Axis/Angle $\rightarrow$ Rotation Matrix $\overset{\mathbf{A}}{\underset{\mathbf{B}}{\mathbf{R}}}$

$$\alpha = \text{ArcTan}\left[\frac{r_y}{r_x}\right];$$

$$\beta = \text{ArcTan}\left[\frac{\sqrt{r_x^2 + r_y^2}}{r_z}\right];$$

$$\overset{\mathbf{A}}{\underset{\mathbf{B}}{\mathbf{R}}} = \mathbf{R}[\gamma, \hat{\mathbf{r}}] = \mathbf{R}_x[\alpha] \cdot \mathbf{R}_y[\beta] \cdot \mathbf{R}_x[\gamma] \cdot \mathbf{R}_y[-\beta] \cdot \mathbf{R}_x[-\alpha]$$

$$= \begin{pmatrix} \text{Cos}[\alpha] & -\text{Sin}[\alpha] & 0 \\ \text{Sin}[\alpha] & \text{Cos}[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}[\beta] & 0 & \text{Sin}[\beta] \\ 0 & 1 & 0 \\ -\text{Sin}[\beta] & 0 & \text{Cos}[\beta] \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}[\gamma] & -\text{Sin}[\gamma] & 0 \\ \text{Sin}[\gamma] & \text{Cos}[\gamma] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}[-\beta] & 0 & \text{Sin}[-\beta] \\ 0 & 1 & 0 \\ -\text{Sin}[-\beta] & 0 & \text{Cos}[-\beta] \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}[-\alpha] & -\text{Sin}[-\alpha] & 0 \\ \text{Sin}[-\alpha] & \text{Cos}[-\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} r_x^2 (1 - \cos[\gamma]) + \cos[\gamma] & r_x r_y (1 - \cos[\gamma]) - r_s \sin[\gamma] & r_x r_s (1 - \cos[\gamma]) + r_y \sin[\gamma] \\ r_x r_y (1 - \cos[\gamma]) + r_s \sin[\gamma] & r_y^2 (1 - \cos[\gamma]) + \cos[\gamma] & r_y r_s (1 - \cos[\gamma]) - r_x \sin[\gamma] \\ r_x r_s (1 - \cos[\gamma]) - r_y \sin[\gamma] & r_y r_s (1 - \cos[\gamma]) + r_x \sin[\gamma] & r_s^2 (1 - \cos[\gamma]) + \cos[\gamma] \end{pmatrix}$$

$$\gamma = \text{ArcCos}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right]$$

$$\hat{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \frac{1}{2 \sin[\gamma]} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

- Orientation $\overset{A}{B} \hat{q} \rightarrow \text{Axis/Angle} \rightarrow \text{Rotation Matrix } \overset{A}{B} R$

$$\cos\left[\frac{\gamma}{2}\right] = \pm \sqrt{\frac{1 + \cos[\gamma]}{2}} \quad \sin\left[\frac{\gamma}{2}\right] = \pm \sqrt{\frac{1 - \cos[\gamma]}{2}}$$

$$\overset{A}{B} \hat{q} = (q_1 \quad q_2 \quad q_3 \quad q_4) = \left(\cos\left[\frac{\gamma}{2}\right] \quad r_x \sin\left[\frac{\gamma}{2}\right] \quad r_y \sin\left[\frac{\gamma}{2}\right] \quad r_s \sin\left[\frac{\gamma}{2}\right] \right)$$

- 1

$$q_1 = \cos\left[\frac{\gamma}{2}\right] = \pm \sqrt{\frac{1 + \cos[\gamma]}{2}}$$

$$q_2 = r_x \sin\left[\frac{\gamma}{2}\right] = \pm r_x \sqrt{\frac{1 - \cos[\gamma]}{2}}$$

$$q_3 = r_y \sin\left[\frac{\gamma}{2}\right] = \pm r_y \sqrt{\frac{1 - \cos[\gamma]}{2}}$$

$$q_4 = r_s \sin\left[\frac{\gamma}{2}\right] = \pm r_s \sqrt{\frac{1 - \cos[\gamma]}{2}}$$

- 2

$$q_1^2 = \frac{1 + \cos[\gamma]}{2}$$

$$q_2^2 = r_x^2 \frac{1 - \cos[\gamma]}{2}$$

$$q_3^2 = r_y^2 \frac{1 - \cos[\gamma]}{2}$$

$$q_4^2 = r_s^2 \frac{1 - \cos[\gamma]}{2}$$

- 3

$$\cos[\gamma] = 2 q_1^2 - 1$$

$$r_x^2 (1 - \cos[\gamma]) = 2 q_2^2$$

$$r_y^2 (1 - \cos[\gamma]) = 2 q_3^2$$

$$r_s^2 (1 - \cos[\gamma]) = 2 q_4^2$$

$$\sin[\gamma] = \sqrt{1 - \cos[\gamma]^2} = \sqrt{(1 - \cos[\gamma]) (1 + \cos[\gamma])} = \sqrt{2 q_1^2 (1 - \cos[\gamma])}$$

$$\sin[\gamma] = \sqrt{2} q_1 \sqrt{1 - \cos[\gamma]}$$

■ 4

$$r_x = \frac{\sqrt{2} q_2}{\sqrt{1 - \cos[\gamma]}}$$

$$r_y = \frac{\sqrt{2} q_3}{\sqrt{1 - \cos[\gamma]}}$$

$$r_s = \frac{\sqrt{2} q_4}{\sqrt{1 - \cos[\gamma]}}$$

$$\sin[\gamma] = \sqrt{2} q_1 \sqrt{1 - \cos[\gamma]}$$

■ 5

$$\mathbf{A}_B \mathbf{R} = \begin{pmatrix} r_x^2 (1 - \cos[\gamma]) + \cos[\gamma] & r_x r_y (1 - \cos[\gamma]) - r_s \sin[\gamma] & r_x r_s (1 - \cos[\gamma]) + r_y \sin[\gamma] \\ r_x r_y (1 - \cos[\gamma]) + r_s \sin[\gamma] & r_y^2 (1 - \cos[\gamma]) + \cos[\gamma] & r_y r_s (1 - \cos[\gamma]) - r_x \sin[\gamma] \\ r_x r_s (1 - \cos[\gamma]) - r_y \sin[\gamma] & r_y r_s (1 - \cos[\gamma]) + r_x \sin[\gamma] & r_s^2 (1 - \cos[\gamma]) + \cos[\gamma] \end{pmatrix}$$

$$\left(\begin{array}{ccc} 2 q_2^2 + 2 q_1^2 - 1 & \frac{\sqrt{2} q_2}{\sqrt{1 - \cos[\gamma]}} \frac{\sqrt{2} q_3}{\sqrt{1 - \cos[\gamma]}} (1 - \cos[\gamma]) - \frac{\sqrt{2} q_4}{\sqrt{1 - \cos[\gamma]}} \sqrt{2} q_1 \sqrt{1 - \cos[\gamma]} & \frac{\sqrt{2} q_2}{\sqrt{1 - \cos[\gamma]}} \frac{\sqrt{2} q_4}{\sqrt{1 - \cos[\gamma]}} (1 - \cos[\gamma]) + \frac{\sqrt{2} q_3}{\sqrt{1 - \cos[\gamma]}} \sqrt{2} q_1 \sqrt{1 - \cos[\gamma]} \\ \frac{\sqrt{2} q_2}{\sqrt{1 - \cos[\gamma]}} \frac{\sqrt{2} q_3}{\sqrt{1 - \cos[\gamma]}} (1 - \cos[\gamma]) + \frac{\sqrt{2} q_4}{\sqrt{1 - \cos[\gamma]}} \sqrt{2} q_1 \sqrt{1 - \cos[\gamma]} & 2 q_3^2 + 2 q_1^2 - 1 & \frac{\sqrt{2} q_3}{\sqrt{1 - \cos[\gamma]}} \frac{\sqrt{2} q_4}{\sqrt{1 - \cos[\gamma]}} (1 - \cos[\gamma]) - \frac{\sqrt{2} q_2}{\sqrt{1 - \cos[\gamma]}} \sqrt{2} q_1 \sqrt{1 - \cos[\gamma]} \\ \frac{\sqrt{2} q_2}{\sqrt{1 - \cos[\gamma]}} \frac{\sqrt{2} q_4}{\sqrt{1 - \cos[\gamma]}} (1 - \cos[\gamma]) - \frac{\sqrt{2} q_3}{\sqrt{1 - \cos[\gamma]}} \sqrt{2} q_1 \sqrt{1 - \cos[\gamma]} & \frac{\sqrt{2} q_3}{\sqrt{1 - \cos[\gamma]}} \frac{\sqrt{2} q_4}{\sqrt{1 - \cos[\gamma]}} (1 - \cos[\gamma]) + \frac{\sqrt{2} q_2}{\sqrt{1 - \cos[\gamma]}} \sqrt{2} q_1 \sqrt{1 - \cos[\gamma]} & 2 q_4^2 + 2 q_1^2 - 1 \end{array} \right) // \text{FullSimplify} // \text{Together} //$$

MatrixForm

$$\mathbf{A}_B \mathbf{R} = \begin{pmatrix} -1 + 2 q_1^2 + 2 q_2^2 & 2 (q_2 q_3 - q_1 q_4) & 2 (q_1 q_3 + q_2 q_4) \\ 2 (q_2 q_3 + q_1 q_4) & -1 + 2 q_1^2 + 2 q_3^2 & -2 (q_1 q_2 - q_3 q_4) \\ -2 (q_1 q_3 - q_2 q_4) & 2 (q_1 q_2 + q_3 q_4) & -1 + 2 q_1^2 + 2 q_4^2 \end{pmatrix}$$

■ 6

$$(q_1 \ q_2 \ q_3 \ q_4) = (\eta \ \epsilon_x \ \epsilon_y \ \epsilon_s);$$

$$\left(\begin{array}{ccc} -1 + 2 q_1^2 + 2 q_2^2 & 2 (q_2 q_3 - q_1 q_4) & 2 (q_1 q_3 + q_2 q_4) \\ 2 (q_2 q_3 + q_1 q_4) & -1 + 2 q_1^2 + 2 q_3^2 & -2 (q_1 q_2 - q_3 q_4) \\ -2 (q_1 q_3 - q_2 q_4) & 2 (q_1 q_2 + q_3 q_4) & -1 + 2 q_1^2 + 2 q_4^2 \end{array} \right) // \text{MatrixForm}$$

$$\mathbf{R}[\eta, \epsilon] = \begin{pmatrix} 2 (\eta^2 + \epsilon_x^2) - 1 & 2 (\epsilon_x \epsilon_y - \eta \epsilon_s) & 2 (\epsilon_x \epsilon_s + \eta \epsilon_y) \\ 2 (\epsilon_x \epsilon_y + \eta \epsilon_s) & 2 (\eta^2 + \epsilon_y^2) - 1 & 2 (\epsilon_y \epsilon_s - \eta \epsilon_x) \\ 2 (\epsilon_x \epsilon_s - \eta \epsilon_y) & 2 (\epsilon_y \epsilon_s + \eta \epsilon_x) & 2 (\eta^2 + \epsilon_s^2) - 1 \end{pmatrix}$$

Orientation ${}^A_B \hat{q} \rightarrow \text{Roll } \phi \hat{x}_B, \text{Pitch } \theta \hat{y}_B, \text{Yaw } \psi \hat{z}_B$

ϕ around $\hat{x}_B$, θ around $\hat{y}_B$, ψ around $\hat{z}_B$

$${}^A_B R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} -1 + 2 q_1^2 + 2 q_2^2 & 2 (q_2 q_3 - q_1 q_4) & 2 (q_1 q_3 + q_2 q_4) \\ 2 (q_2 q_3 + q_1 q_4) & -1 + 2 q_1^2 + 2 q_3^2 & -2 (q_1 q_2 - q_3 q_4) \\ -2 (q_1 q_3 - q_2 q_4) & 2 (q_1 q_2 + q_3 q_4) & -1 + 2 q_1^2 + 2 q_4^2 \end{pmatrix}$$

$$\phi = \text{ArcTan} \left[{}^A_B R_{33}, {}^A_B R_{32} \right] = \text{ArcTan} \left[2 q_1^2 - 1 + 2 q_4^2, 2 (q_3 q_4 - q_1 q_2) \right]$$

$$\theta = -\text{ArcTan} \left[\sqrt{1 - {}^A_B R_{31}^2}, {}^A_B R_{31} \right] = \text{ArcSin} \left[-{}^A_B R_{31} \right] = -\text{ArcSin} \left[2 (q_2 q_4 + q_1 q_3) \right]$$

$$\psi = \text{ArcTan} \left[{}^A_B R_{11}, {}^A_B R_{12} \right] = \text{ArcTan} \left[2 q_1^2 - 1 + 2 q_2^2, 2 (q_2 q_3 - q_1 q_4) \right]$$

Roll $\phi \hat{x}_B$, Pitch $\theta \hat{y}_B$, Yaw $\psi \hat{z}_B$

$\vdots$

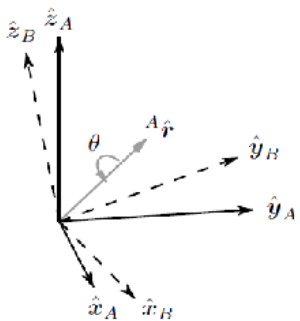
Rotation Matrix ${}^A_B R$

... Axis / Angle

$\vdots$

Quaternion ${}^A_B \hat{q}$

Axis/Angle



- A rotation of angle θ around an axis ${}^A \hat{r}$ defined in frame A

$${}^A \hat{r} = (r_x \ r_y \ r_z)$$

$${}^A\mathbf{P} = \begin{pmatrix} {}^A\mathbf{p}_x \\ {}^A\mathbf{p}_Y \\ {}^A\mathbf{p}_z \end{pmatrix}, \quad {}^B\mathbf{P} = \begin{pmatrix} {}^B\mathbf{p}_x \\ {}^B\mathbf{p}_Y \\ {}^B\mathbf{p}_z \end{pmatrix}$$

$$\hat{\mathbf{x}}_A = \begin{pmatrix} {}^A\mathbf{x}_x \\ {}^A\mathbf{x}_Y \\ {}^A\mathbf{x}_z \end{pmatrix}, \quad \hat{\mathbf{y}}_A = \begin{pmatrix} {}^A\mathbf{y}_x \\ {}^A\mathbf{y}_Y \\ {}^A\mathbf{y}_z \end{pmatrix}, \quad \hat{\mathbf{z}}_A = \begin{pmatrix} {}^A\mathbf{z}_x \\ {}^A\mathbf{z}_Y \\ {}^A\mathbf{z}_z \end{pmatrix}$$

$$\hat{\mathbf{x}}_B = \begin{pmatrix} {}^B\mathbf{x}_x \\ {}^B\mathbf{x}_Y \\ {}^B\mathbf{x}_z \end{pmatrix}, \quad \hat{\mathbf{y}}_B = \begin{pmatrix} {}^B\mathbf{y}_x \\ {}^B\mathbf{y}_Y \\ {}^B\mathbf{y}_z \end{pmatrix}, \quad \hat{\mathbf{z}}_B = \begin{pmatrix} {}^B\mathbf{z}_x \\ {}^B\mathbf{z}_Y \\ {}^B\mathbf{z}_z \end{pmatrix}$$

$$\begin{pmatrix} \hat{\mathbf{x}}_A \\ \hat{\mathbf{y}}_A \\ \hat{\mathbf{z}}_A \end{pmatrix}, \quad \begin{pmatrix} \hat{\mathbf{x}}_B \\ \hat{\mathbf{y}}_B \\ \hat{\mathbf{z}}_B \end{pmatrix}$$

$${}^A\mathbf{P} = {}^A_B\mathbf{R} {}^B\mathbf{P}$$

$$\begin{aligned} \mathbf{P} &= {}^A\mathbf{P} = {}^A\mathbf{p}_x \hat{\mathbf{z}}_A + {}^A\mathbf{p}_Y \hat{\mathbf{y}}_A + {}^A\mathbf{p}_z \hat{\mathbf{z}}_A \\ &= {}^B\mathbf{P} = {}^B\mathbf{p}_x \hat{\mathbf{z}}_B + {}^B\mathbf{p}_Y \hat{\mathbf{y}}_B + {}^B\mathbf{p}_z \hat{\mathbf{z}}_B \end{aligned}$$

$${}^A\mathbf{P} = \begin{pmatrix} {}^A\mathbf{p}_x \\ {}^A\mathbf{p}_Y \\ {}^A\mathbf{p}_z \end{pmatrix} = \begin{pmatrix} {}^A\mathbf{P} \hat{\mathbf{z}}_A \\ {}^A\mathbf{P} \hat{\mathbf{y}}_A \\ {}^A\mathbf{P} \hat{\mathbf{z}}_A \end{pmatrix} = \begin{pmatrix} {}^B\mathbf{P} \hat{\mathbf{z}}_A \\ {}^B\mathbf{P} \hat{\mathbf{y}}_A \\ {}^B\mathbf{P} \hat{\mathbf{z}}_A \end{pmatrix} = \begin{pmatrix} ({}^B\mathbf{p}_x \hat{\mathbf{z}}_B + {}^B\mathbf{p}_Y \hat{\mathbf{y}}_B + {}^B\mathbf{p}_z \hat{\mathbf{z}}_B) \hat{\mathbf{z}}_A \\ ({}^B\mathbf{p}_x \hat{\mathbf{z}}_B + {}^B\mathbf{p}_Y \hat{\mathbf{y}}_B + {}^B\mathbf{p}_z \hat{\mathbf{z}}_B) \hat{\mathbf{y}}_A \\ ({}^B\mathbf{p}_x \hat{\mathbf{z}}_B + {}^B\mathbf{p}_Y \hat{\mathbf{y}}_B + {}^B\mathbf{p}_z \hat{\mathbf{z}}_B) \hat{\mathbf{z}}_A \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{z}}_B \hat{\mathbf{z}}_A & \hat{\mathbf{y}}_B \hat{\mathbf{z}}_A & \hat{\mathbf{z}}_B \hat{\mathbf{z}}_A \\ \hat{\mathbf{z}}_B \hat{\mathbf{y}}_A & \hat{\mathbf{y}}_B \hat{\mathbf{y}}_A & \hat{\mathbf{z}}_B \hat{\mathbf{y}}_A \\ \hat{\mathbf{z}}_B \hat{\mathbf{z}}_A & \hat{\mathbf{y}}_B \hat{\mathbf{z}}_A & \hat{\mathbf{z}}_B \hat{\mathbf{z}}_A \end{pmatrix} \cdot \begin{pmatrix} {}^B\mathbf{p}_x \\ {}^B\mathbf{p}_Y \\ {}^B\mathbf{p}_z \end{pmatrix} = {}^A_B\mathbf{R} {}^B\mathbf{P}$$

$${}^A_B\mathbf{P} = \begin{pmatrix} \hat{\mathbf{z}}_B \hat{\mathbf{z}}_A & \hat{\mathbf{y}}_B \hat{\mathbf{z}}_A & \hat{\mathbf{z}}_B \hat{\mathbf{z}}_A \\ \hat{\mathbf{z}}_B \hat{\mathbf{y}}_A & \hat{\mathbf{y}}_B \hat{\mathbf{y}}_A & \hat{\mathbf{z}}_B \hat{\mathbf{y}}_A \\ \hat{\mathbf{z}}_B \hat{\mathbf{z}}_A & \hat{\mathbf{y}}_B \hat{\mathbf{z}}_A & \hat{\mathbf{z}}_B \hat{\mathbf{z}}_A \end{pmatrix} \cdot \begin{pmatrix} {}^B\mathbf{p}_x \\ {}^B\mathbf{p}_Y \\ {}^B\mathbf{p}_z \end{pmatrix} = \begin{pmatrix} (\hat{\mathbf{z}}_B \hat{\mathbf{z}}_A) {}^B\mathbf{p}_x + (\hat{\mathbf{y}}_B \hat{\mathbf{z}}_A) {}^B\mathbf{p}_Y + (\hat{\mathbf{z}}_B \hat{\mathbf{z}}_A) {}^B\mathbf{p}_z \\ (\hat{\mathbf{z}}_B \hat{\mathbf{y}}_A) {}^B\mathbf{p}_x + (\hat{\mathbf{y}}_B \hat{\mathbf{y}}_A) {}^B\mathbf{p}_Y + (\hat{\mathbf{z}}_B \hat{\mathbf{y}}_A) {}^B\mathbf{p}_z \\ (\hat{\mathbf{z}}_B \hat{\mathbf{z}}_A) {}^B\mathbf{p}_x + (\hat{\mathbf{y}}_B \hat{\mathbf{z}}_A) {}^B\mathbf{p}_Y + (\hat{\mathbf{z}}_B \hat{\mathbf{z}}_A) {}^B\mathbf{p}_z \end{pmatrix} = \begin{pmatrix} ({}^B\mathbf{p}_x \hat{\mathbf{z}}_B + {}^B\mathbf{p}_Y \hat{\mathbf{y}}_B + {}^B\mathbf{p}_z \hat{\mathbf{z}}_B) \hat{\mathbf{z}}_A \\ ({}^B\mathbf{p}_x \hat{\mathbf{z}}_B + {}^B\mathbf{p}_Y \hat{\mathbf{y}}_B + {}^B\mathbf{p}_z \hat{\mathbf{z}}_B) \hat{\mathbf{y}}_A \\ ({}^B\mathbf{p}_x \hat{\mathbf{z}}_B + {}^B\mathbf{p}_Y \hat{\mathbf{y}}_B + {}^B\mathbf{p}_z \hat{\mathbf{z}}_B) \hat{\mathbf{z}}_A \end{pmatrix} = \begin{pmatrix} {}^B\mathbf{P} \hat{\mathbf{z}}_A \\ {}^B\mathbf{P} \hat{\mathbf{y}}_A \\ {}^B\mathbf{P} \hat{\mathbf{z}}_A \end{pmatrix} = \begin{pmatrix} {}^A\mathbf{P} \hat{\mathbf{z}}_A \\ {}^A\mathbf{P} \hat{\mathbf{y}}_A \\ {}^A\mathbf{P} \hat{\mathbf{z}}_A \end{pmatrix} = \begin{pmatrix} {}^A\mathbf{p}_x \\ {}^A\mathbf{p}_Y \\ {}^A\mathbf{p}_z \end{pmatrix} = {}^A\mathbf{P}$$

$${}^A_B\mathbf{R} = \begin{pmatrix} \hat{\mathbf{x}}_B \hat{\mathbf{x}}_A & \hat{\mathbf{y}}_B \hat{\mathbf{x}}_A & \hat{\mathbf{z}}_B \hat{\mathbf{x}}_A \\ \hat{\mathbf{x}}_B \hat{\mathbf{y}}_A & \hat{\mathbf{y}}_B \hat{\mathbf{y}}_A & \hat{\mathbf{z}}_B \hat{\mathbf{y}}_A \\ \hat{\mathbf{x}}_B \hat{\mathbf{z}}_A & \hat{\mathbf{y}}_B \hat{\mathbf{z}}_A & \hat{\mathbf{z}}_B \hat{\mathbf{z}}_A \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

$$\text{Axis/Angle} \rightarrow \text{Rotation Matrix } {}^A_B\mathbf{R}$$

$$\alpha = \text{ArcTan}\left[\frac{r_Y}{r_x}\right];$$

$$\beta = \text{ArcTan}\left[\frac{\sqrt{r_x^2 + r_Y^2}}{r_z}\right];$$

$$\mathbf{R}[\gamma, \mathbf{r}] = \mathbf{R}_z[\alpha] \cdot \mathbf{R}_Y[\beta] \cdot \mathbf{R}_z[\gamma] \cdot \mathbf{R}_Y[-\beta] \cdot \mathbf{R}_z[-\alpha]$$

$$= \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[\gamma] & -\sin[\gamma] & 0 \\ \sin[\gamma] & \cos[\gamma] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[-\beta] & 0 & \sin[-\beta] \\ 0 & 1 & 0 \\ -\sin[-\beta] & 0 & \cos[-\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[-\alpha] & -\sin[-\alpha] & 0 \\ \sin[-\alpha] & \cos[-\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$${}^A_B\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} r_x^2 (1 - \cos[\gamma]) + \cos[\gamma] & r_x r_Y (1 - \cos[\gamma]) - r_z \sin[\gamma] & r_x r_z (1 - \cos[\gamma]) + r_Y \sin[\gamma] \\ r_x r_Y (1 - \cos[\gamma]) + r_z \sin[\gamma] & r_Y^2 (1 - \cos[\gamma]) + \cos[\gamma] & r_Y r_z (1 - \cos[\gamma]) - r_x \sin[\gamma] \\ r_x r_z (1 - \cos[\gamma]) - r_Y \sin[\gamma] & r_Y r_z (1 - \cos[\gamma]) + r_x \sin[\gamma] & r_z^2 (1 - \cos[\gamma]) + \cos[\gamma] \end{pmatrix}$$

- **Axis/Angle** $\rightarrow$ Roll ϕ $\hat{x}_B$, Pitch θ $\hat{y}_B$, Yaw ψ $\hat{z}_B$, ZYX

ϕ around $\hat{x}_B$, θ around $\hat{y}_B$, ψ around $\hat{z}_B$

$$\phi \text{ (Roll)} = \text{ArcTan}[r_{33}, r_{32}] = \text{ArcTan}[r_x^2 (1 - \cos[\gamma]) + \cos[\gamma], r_y r_x (1 - \cos[\gamma]) + r_x \sin[\gamma]]$$

$$\theta \text{ (Pitch)} = \text{ArcSin}[-r_{31}] = \text{ArcSin}[r_y \sin[\gamma] - r_x r_z (1 - \cos[\gamma])]$$

$$\psi \text{ (Yaw)} = \text{ArcTan}[r_{11}, r_{12}] = \text{ArcTan}[r_x^2 (1 - \cos[\gamma]) + \cos[\gamma], r_x r_y (1 - \cos[\gamma]) + r_x \sin[\gamma]]$$

- **Axis/Angle** $\rightarrow$ Quaternion ${}^A_B \hat{q}$

- An arbitrary orientation of frame B relative to frame A

$${}^A_B \hat{q} = (q_1 \quad q_2 \quad q_3 \quad q_4) = \left(\cos\left[\frac{\gamma}{2}\right] \quad -r_x \sin\left[\frac{\gamma}{2}\right] \quad -r_y \sin\left[\frac{\gamma}{2}\right] \quad -r_z \sin\left[\frac{\gamma}{2}\right] \right)$$

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

Euler Angle (Roll ϕ $\hat{x}_B$, Pitch θ $\hat{y}_B$, Yaw ψ $\hat{z}_B$, ZYX) $\rightarrow$ Rotation Matrix ${}^A_B R$

X axis (Roll) $\phi \rightarrow R_x[\phi] \rightarrow \{-\pi, \pi\}$

Y axis (Pitch) $\theta \rightarrow R_y[\theta] \rightarrow \{-\pi/2, \pi/2\}$

Z axis (Yaw) $\psi \rightarrow R_z[\psi] \rightarrow \{-\pi, \pi\}$

$${}^A_B R = R_{zyx} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

$$= R_x[\psi] \cdot R_y[\theta] \cdot R_z[\phi]$$

$$= \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$= \begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

- **Euler Angle (Roll ϕ $\hat{x}_B$, Pitch θ $\hat{y}_B$, Yaw ψ $\hat{z}_B$, ZYX) $\rightarrow$ Axis/Angle**

$$\gamma = \text{ArcCos}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right] = \text{ArcCos}\left[\frac{\cos[\theta] \cos[\psi] + \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] + \cos[\theta] \cos[\phi] - 1}{2}\right]$$

$$\hat{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \frac{1}{2 \sin[\gamma]} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix} = \frac{1}{2 \sin[\gamma]} \begin{pmatrix} (\cos[\theta] \sin[\phi]) - (-\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi]) \\ (\cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi]) - (-\sin[\theta]) \\ (\cos[\theta] \sin[\psi]) - (\cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi]) \end{pmatrix}$$

- **Euler Angle (Roll ϕ $\hat{x}_B$, Pitch θ $\hat{y}_B$, Yaw ψ $\hat{z}_B$, ZYX) $\rightarrow$ Quaternion ${}^A_B \hat{q}$**

$$q_1 = \frac{1}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1}$$

$$q_2 = -\frac{\text{Sign}[r_{32} - r_{23}]}{2} \sqrt{r_{11} - r_{22} - r_{33} + 1}$$

$$q_3 = -\frac{\text{Sign}[r_{13} - r_{31}]}{2} \sqrt{r_{22} - r_{33} - r_{11} + 1}$$

$$q_4 = -\frac{\text{Sign}[r_{21} - r_{12}]}{2} \sqrt{r_{33} - r_{11} - r_{22} + 1}$$

$$q_1 = \frac{1}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1}$$

$$= \frac{1}{2} \sqrt{1 + \text{Cos}[\theta] \text{Cos}[\phi] + \text{Cos}[\theta] \text{Cos}[\psi] + \text{Cos}[\phi] \text{Cos}[\psi] + \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi]}$$

$$q_2 = -\frac{\text{Sign}[r_{32} - r_{23}]}{2} \sqrt{r_{11} - r_{22} - r_{33} + 1}$$

$$= -\frac{1}{2} \text{Sign}[\text{Cos}[\theta] \text{Sin}[\phi] + \text{Cos}[\psi] \text{Sin}[\phi] - \text{Cos}[\phi] \text{Sin}[\theta] \text{Sin}[\psi]] \sqrt{1 - \text{Cos}[\theta] \text{Cos}[\phi] + \text{Cos}[\theta] \text{Cos}[\psi] - \text{Cos}[\phi] \text{Cos}[\psi] - \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi]}$$

$$q_3 = -\frac{\text{Sign}[r_{13} - r_{31}]}{2} \sqrt{r_{22} - r_{33} - r_{11} + 1}$$

$$= -\frac{1}{2} \text{Sign}[\text{Sin}[\theta] + \text{Cos}[\phi] \text{Cos}[\psi] \text{Sin}[\theta] + \text{Sin}[\phi] \text{Sin}[\psi]] \sqrt{1 - \text{Cos}[\theta] \text{Cos}[\phi] - \text{Cos}[\theta] \text{Cos}[\psi] + \text{Cos}[\phi] \text{Cos}[\psi] + \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi]}$$

$$q_4 = -\frac{\text{Sign}[r_{21} - r_{12}]}{2} \sqrt{r_{33} - r_{11} - r_{22} + 1}$$

$$= -\frac{1}{2} \text{Sign}[-\text{Cos}[\psi] \text{Sin}[\theta] \text{Sin}[\phi] + \text{Cos}[\theta] \text{Sin}[\psi] + \text{Cos}[\phi] \text{Sin}[\psi]] \sqrt{1 + \text{Cos}[\theta] \text{Cos}[\phi] - \text{Cos}[\theta] \text{Cos}[\psi] - \text{Cos}[\phi] \text{Cos}[\psi] - \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi]}$$

Quaternion $\overset{\text{A}}{\text{B}} \hat{\mathbf{Q}} \rightarrow \text{Rotation Matrix } \overset{\text{A}}{\text{B}} \mathbf{R}$

$$\overset{\text{A}}{\text{B}} \hat{\mathbf{Q}} = (q_1 \quad q_2 \quad q_3 \quad q_4) = \left(\text{Cos}\left[\frac{\gamma}{2}\right] \quad -r_x \text{Sin}\left[\frac{\gamma}{2}\right] \quad -r_y \text{Sin}\left[\frac{\gamma}{2}\right] \quad -r_z \text{Sin}\left[\frac{\gamma}{2}\right] \right)$$

$$\text{Cos}\left[\frac{\gamma}{2}\right] = \pm \sqrt{\frac{1 + \text{Cos}[\gamma]}{2}} \quad , \quad \text{Sin}\left[\frac{\gamma}{2}\right] = \pm \sqrt{\frac{1 - \text{Cos}[\gamma]}{2}}$$

$$\overset{\text{A}}{\text{B}} \mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

$$= \begin{pmatrix} r_x^2 (1 - \text{Cos}[\theta]) + \text{Cos}[\theta] & r_x r_y (1 - \text{Cos}[\theta]) - r_z \text{Sin}[\theta] & r_x r_z (1 - \text{Cos}[\theta]) + r_y \text{Sin}[\theta] \\ r_x r_y (1 - \text{Cos}[\theta]) + r_z \text{Sin}[\theta] & r_y^2 (1 - \text{Cos}[\theta]) + \text{Cos}[\theta] & r_y r_z (1 - \text{Cos}[\theta]) - r_x \text{Sin}[\theta] \\ r_x r_z (1 - \text{Cos}[\theta]) - r_y \text{Sin}[\theta] & r_y r_z (1 - \text{Cos}[\theta]) + r_x \text{Sin}[\theta] & r_z^2 (1 - \text{Cos}[\theta]) + \text{Cos}[\theta] \end{pmatrix}$$

$$= \begin{pmatrix} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 + q_1 q_4) & 2 (q_2 q_4 - q_1 q_3) \\ 2 (q_2 q_3 - q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 + q_1 q_2) \\ 2 (q_2 q_4 + q_1 q_3) & 2 (q_3 q_4 - q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{pmatrix}$$

- Quaternion ${}^A_B \hat{q} \rightarrow$ Euler Angle (Roll ϕ $\hat{x}_B$, Pitch θ $\hat{y}_B$, Yaw ψ $\hat{z}_B$, ZYX)

$$\phi = \text{ArcTan}\left[{}^A_B R_{33}, {}^A_B R_{32}\right] = \text{ArcTan}\left[2 q_1^2 - 1 + 2 q_4^2, 2 (q_3 q_4 - q_1 q_2)\right]$$

$$\theta = \text{ArcSin}\left[-{}^A_B R_{31}\right] = -\text{ArcSin}\left[2 (q_2 q_4 + q_1 q_3)\right]$$

$$\psi = \text{ArcTan}\left[{}^A_B R_{11}, {}^A_B R_{12}\right] = \text{ArcTan}\left[2 q_1^2 - 1 + 2 q_2^2, 2 (q_2 q_3 - q_1 q_4)\right]$$

- Quaternion ${}^A_B \hat{q} \rightarrow$ Axis/Angle

- 1

$$\begin{aligned} \gamma &= \text{ArcCos}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right] \\ &= \text{ArcCos}\left[\frac{(2 q_1^2 - 1 + 2 q_2^2) + (2 q_1^2 - 1 + 2 q_3^2) + (2 q_1^2 - 1 + 2 q_4^2) - 1}{2}\right] \\ &= \text{ArcCos}\left[-2 + 3 q_1^2 + q_2^2 + q_3^2 + q_4^2\right] \\ &= \text{ArcCos}\left[2 q_1^2 - 1\right] \end{aligned}$$

- 2

$$\begin{aligned} \hat{r} &= \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \frac{1}{2 \sin[\gamma]} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix} \\ &= \frac{1}{2 \sin[\gamma]} \begin{pmatrix} 2 (q_3 q_4 - q_1 q_2) - 2 (q_3 q_4 + q_1 q_2) \\ 2 (q_2 q_4 - q_1 q_3) - 2 (q_2 q_4 + q_1 q_3) \\ 2 (q_2 q_3 - q_1 q_4) - 2 (q_2 q_3 + q_1 q_4) \end{pmatrix} \\ &= \frac{1}{\sin[\gamma]} \begin{pmatrix} -2 q_1 q_2 \\ -2 q_1 q_3 \\ -2 q_1 q_4 \end{pmatrix} \end{aligned}$$

- 3

$$\begin{aligned} q_1 &= \cos\left[\frac{\gamma}{2}\right] = \pm \sqrt{\frac{1 + \cos[\gamma]}{2}} \\ \sin\left[\frac{\gamma}{2}\right] &= \pm \sqrt{\frac{1 - \cos[\gamma]}{2}} = \pm \sqrt{1 - q_1^2} \\ \sin[\gamma] &= 2 \sin\left[\frac{\gamma}{2}\right] \cos\left[\frac{\gamma}{2}\right] = 2 q_1 \sqrt{1 - q_1^2} \end{aligned}$$

- 4

$$\begin{aligned} {}^A_B \hat{q} &= (q_1 \quad -r_x \sqrt{1 - q_1^2} \quad -r_y \sqrt{1 - q_1^2} \quad -r_z \sqrt{1 - q_1^2}) \\ \gamma &= 2 \text{ArcCos}[q_1] = \text{ArcCos}[2 q_1^2 - 1] \end{aligned}$$

$$\begin{aligned}\hat{\mathbf{r}} &= \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \frac{1}{\sin[\gamma]} \begin{pmatrix} -2 \, q_1 \, q_2 \\ -2 \, q_1 \, q_3 \\ -2 \, q_1 \, q_4 \end{pmatrix} \\ &= \frac{1}{2 \, q_1 \sqrt{1 - q_1^2}} \begin{pmatrix} -2 \, q_1 \, q_2 \\ -2 \, q_1 \, q_3 \\ -2 \, q_1 \, q_4 \end{pmatrix} \\ &= \frac{-1}{\sqrt{1 - q_1^2}} \begin{pmatrix} q_2 \\ q_3 \\ q_4 \end{pmatrix}\end{aligned}$$

▪ 5

$${}^A\hat{\mathbf{q}} = (\, q_1 \, \, q_2 \, \, q_3 \, \, q_4 \,) = (\, q_1 \, \, -r_x \sqrt{1 - q_1^2} \, \, -r_y \sqrt{1 - q_1^2} \, \, -r_z \sqrt{1 - q_1^2} \,)$$

$$\gamma = \text{ArcCos}\big[2 \, q_1^2 - 1\big]$$

$$\hat{\mathbf{r}} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \frac{-1}{\sqrt{1 - q_1^2}} \begin{pmatrix} q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

$$\text{ArcCos}\big[2 \, q_1^2 - 1\big] \, // \, \text{InputForm}$$

$$\text{ArcCos}[-1 + 2^*q1^2]$$

$$\frac{-1}{\sqrt{1 - q1^2}} \begin{pmatrix} q2 \\ q3 \\ q4 \end{pmatrix} \, // \, \text{InputForm}$$

$$\{\{-q2/\text{Sqrt}[1 - q1^2]\}, \{-q3/\text{Sqrt}[1 - q1^2]\}, \{-q4/\text{Sqrt}[1 - q1^2]\}\}$$

▪ 6

$$\gamma = \text{ArcCos}\big[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\big] = \text{ArcCos}\big[-2 + 3 \, q_1^2 + q_2^2 + q_3^2 + q_4^2\big] = \text{ArcCos}\big[2 \, q_1^2 - 1\big]$$

$$\hat{\mathbf{r}} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \frac{1}{2 \, \sin[\gamma]} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix} = \frac{1}{2 \, \sin[\gamma]} \begin{pmatrix} 2 \, (q_3 \, q_4 - q_1 \, q_2) - 2 \, (q_3 \, q_4 + q_1 \, q_2) \\ 2 \, (q_2 \, q_4 - q_1 \, q_3) - 2 \, (q_2 \, q_4 + q_1 \, q_3) \\ 2 \, (q_2 \, q_3 - q_1 \, q_4) - 2 \, (q_2 \, q_3 + q_1 \, q_4) \end{pmatrix} = \frac{1}{\sin[\gamma]} \begin{pmatrix} -2 \, q_1 \, q_2 \\ -2 \, q_1 \, q_3 \\ -2 \, q_1 \, q_4 \end{pmatrix} = \frac{-1}{\sqrt{1 - q_1^2}} \begin{pmatrix} q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

Rotation Matrix ${}^A_{\text{B}}\mathbf{R} \rightarrow$ Euler Angle (Roll ϕ $\hat{\mathbf{x}}_{\text{B}}$, Pitch θ $\hat{\mathbf{y}}_{\text{B}}$, Yaw ψ $\hat{\mathbf{z}}_{\text{B}}$, ZYX)

$$\phi = \text{ArcTan}[r_{33}, \, r_{32}]$$

$$\theta = \text{ArcSin}[-r_{31}]$$

$$\psi = \text{ArcTan}[r_{11}, \, r_{21}]$$

▪ $\theta = \{-\pi/2, \pi/2\}$

$$\phi \, (\text{Roll}) = \text{ArcTan}[r_{33}, \, r_{32}]$$

$$\theta \, (\text{Pitch}) = \text{ArcSin}[-r_{31}]$$

$$\psi \text{ (Yaw)} = \text{ArcTan}[r_{11}, r_{21}]$$

$$\theta = \{\pi/2, 3\pi/2\}$$

$$\phi = \text{ArcTan}[-r_{33}, -r_{32}]$$

$$\theta = \text{ArcSin}[-r_{31}]$$

$$\psi = \text{ArcTan}[-r_{11}, -r_{21}]$$

$$\theta \text{ (Pitch)} = \pi/2, \text{Cos}[\theta] \approx 0$$

$$r_{23} + r_{12} = (\text{Sin}[\theta] - 1) \text{Sin}[\phi + \psi]$$

$$r_{13} - r_{22} = (\text{Sin}[\theta] - 1) \text{Cos}[\phi + \psi]$$

$$r_{12} - r_{23} = (\text{Sin}[\theta] + 1) \text{Sin}[\phi - \psi]$$

$$r_{13} + r_{22} = (\text{Sin}[\theta] + 1) \text{Cos}[\phi - \psi]$$

$$\phi + \psi = \text{ArcTan}[r_{13} - r_{22}, r_{23} + r_{12}]$$

$$\theta = \text{ArcSin}[-r_{31}]$$

$$\phi - \psi = \text{ArcTan}[r_{13} + r_{22}, r_{12} - r_{23}]$$

Rotation Matrix $\overset{A}{B} R \rightarrow \text{Axis/Angle}$

$$\gamma = \text{ArcCos}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right]$$

$$\hat{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \frac{1}{2 \text{Sin}[\gamma]} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

$$r_x^2 + r_y^2 + r_z^2 = 1$$

$$\text{Angle } \gamma \text{ and Axis } r$$

$$\text{Sin}[\alpha] = \frac{r_y}{\sqrt{r_x^2 + r_y^2}}, \text{Cos}[\alpha] = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\text{Sin}[\beta] = \frac{\sqrt{r_x^2 + r_y^2}}{\sqrt{r_x^2 + r_y^2 + r_z^2}}, \text{Cos}[\beta] = \frac{r_z}{\sqrt{r_x^2 + r_y^2 + r_z^2}}$$

$$\text{Rotation matrix} \leftrightarrow \text{Euler axis/angles}$$

$$\text{Angle}$$

$$\gamma = \text{ArcCos}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right]$$

■ Euler axis

$$\hat{\mathbf{r}} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}$$

$$r_x = \frac{r_{32} - r_{23}}{2 \sin[\gamma]}$$

$$r_y = \frac{r_{13} - r_{31}}{2 \sin[\gamma]}$$

$$r_z = \frac{r_{21} - r_{12}}{2 \sin[\gamma]}$$

■ Rotation - concisely

$$\mathbf{R} = \mathbf{I} \cdot \cos[\theta] + \sin[\theta] \mathbf{u}_x + (1 - \cos[\theta]) \mathbf{u} \otimes \mathbf{u}$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{u} \otimes \mathbf{u} = \begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{pmatrix}$$

$$\mathbf{u}_x = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_x u_y (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_x u_z (1 - \cos[\theta]) - u_y \sin[\theta] & u_y u_z (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix}$$

$$\mathbf{R}[\theta_] := \begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_y u_x (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_z u_x (1 - \cos[\theta]) - u_y \sin[\theta] & u_z u_y (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix}$$

■ Example

$$\{u_x, u_y, u_z\} = \{0, 0, 1\} : \begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_y u_x (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_z u_x (1 - \cos[\theta]) - u_y \sin[\theta] & u_z u_y (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix} \cdot \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} A_x \cos[\theta] - A_y \sin[\theta] \\ A_y \cos[\theta] + A_x \sin[\theta] \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_y u_x (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_z u_x (1 - \cos[\theta]) - u_y \sin[\theta] & u_z u_y (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix} \cdot \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} // \text{FullSimplify} // \text{InputForm}$$

$$\begin{aligned} & \{ (Ax \cos[\theta] + Az \sin[\theta] \text{Subscript}[u, y] - (-1 + \cos[\theta]) \text{Subscript}[u, x] \cdot \\ & \quad (Ax \text{Subscript}[u, x] + Ay \text{Subscript}[u, y]) - \\ & \quad (Ay \sin[\theta] + Az(-1 + \cos[\theta]) \text{Subscript}[u, x]) \text{Subscript}[u, z]), \\ & \quad (Ay \cos[\theta] - \text{Subscript}[u, x] (Az \sin[\theta] + Ax(-1 + \cos[\theta]) \text{Subscript}[u, y]) + \\ & \quad Ax \sin[\theta] \text{Subscript}[u, z] - (-1 + \cos[\theta]) \text{Subscript}[u, y] \cdot \\ & \quad (Ay \text{Subscript}[u, y] + Az \text{Subscript}[u, z])), \\ & \quad (Az \cos[\theta] + Ay \sin[\theta] \text{Subscript}[u, x] - Ax \sin[\theta] \text{Subscript}[u, y] - \\ & \quad (-1 + \cos[\theta]) (Ax \text{Subscript}[u, x] + Ay \text{Subscript}[u, y]) \text{Subscript}[u, z] - \\ & \quad Az(-1 + \cos[\theta]) \text{Subscript}[u, z]^2) \} \end{aligned}$$

$$R_{\mathbf{x}}[\phi_]:= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$R_{\mathbf{y}}[\theta_]:= \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} // \text{MatrixForm}$$

$$R_{\mathbf{z}}[\psi_]:= \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[y] & -\sin[y] & 0 \\ \sin[y] & \cos[y] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[p] & 0 & \sin[p] \\ 0 & 1 & 0 \\ -\sin[p] & 0 & \cos[p] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[r] & -\sin[r] \\ 0 & \sin[r] & \cos[r] \end{pmatrix} \cdot \begin{pmatrix} W_x \\ W_y \\ W_z \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} W_x \cos[p] \cos[y] + W_y (\cos[y] \sin[p] \sin[r] - \cos[r] \sin[y]) + W_z (\cos[r] \cos[y] \sin[p] + \sin[r] \sin[y]) \\ W_x \cos[p] \sin[y] + W_z (-\cos[y] \sin[r] + \cos[r] \sin[p] \sin[y]) + W_y (\cos[r] \cos[y] + \sin[p] \sin[r] \sin[y]) \\ W_z \cos[p] \cos[r] - W_x \sin[p] + W_y \cos[p] \sin[r] \end{pmatrix}$$

$$\begin{pmatrix} \cos[p] & 0 & \sin[p] \\ 0 & 1 & 0 \\ -\sin[p] & 0 & \cos[p] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[r] & -\sin[r] \\ 0 & \sin[r] & \cos[r] \end{pmatrix} \cdot \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$\{\{V_x \cos[p] + V_z \cos[r] \sin[p] + V_y \sin[p] \sin[r]\}, \{V_y \cos[r] - V_z \sin[r]\}, \{V_z \cos[p] \cos[r] - V_x \sin[p] + V_y \cos[p] \sin[r]\}\}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[r] & -\sin[r] \\ 0 & \sin[r] & \cos[r] \end{pmatrix} \cdot \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix}$$

$$\{\{U_x\}, \{U_y \cos[r] - U_z \sin[r]\}, \{U_z \cos[r] + U_y \sin[r]\}\}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[-r] & -\sin[-r] \\ 0 & \sin[-r] & \cos[-r] \end{pmatrix} \cdot \begin{pmatrix} \cos[-p] & 0 & \sin[-p] \\ 0 & 1 & 0 \\ -\sin[-p] & 0 & \cos[-p] \end{pmatrix} \cdot \begin{pmatrix} \cos[-y] & -\sin[-y] & 0 \\ \sin[-y] & \cos[-y] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X_x \\ X_y \\ X_z \end{pmatrix}$$

$$\begin{aligned} & \{\{X_x \cos[p] \cos[y] - X_z \sin[p] + X_y \cos[p] \sin[y]\}, \{X_z \cos[p] \sin[r] + X_x (\cos[y] \sin[p] \sin[r] - \cos[r] \sin[y]) + X_y (\cos[r] \cos[y] + \sin[p] \sin[r] \sin[y])\}, \\ & \{X_z \cos[p] \cos[r] + X_y (-\cos[y] \sin[r] + \cos[r] \sin[p] \sin[y]) + X_x (\cos[r] \cos[y] \sin[p] + \sin[r] \sin[y])\}\} \end{aligned}$$

Rotation Matrix ${}^A_B \mathbf{R} \rightarrow \text{Quaternion } {}^A_B \hat{\mathbf{q}}$

$${}^A_B \mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 + q_1 q_4) & 2 (q_2 q_4 - q_1 q_3) \\ 2 (q_2 q_3 - q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 + q_1 q_2) \\ 2 (q_2 q_4 + q_1 q_3) & 2 (q_3 q_4 - q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{pmatrix}$$

■ 1

$${}^A_B \hat{\mathbf{q}} = (q_1 \ q_2 \ q_3 \ q_4)$$

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

■ 2

$$r_{11} + r_{22} + r_{33} + 1$$

$$= (2 q_1^2 - 1 + 2 q_2^2) + (2 q_1^2 - 1 + 2 q_3^2) + (2 q_1^2 - 1 + 2 q_4^2) + 1$$

$$= 2 (-1 + 2 q_1^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2) = 2 (-1 + 2 q_1^2 + 1)$$

$$= 4 q_1^2$$

$$r_{11} - r_{22} - r_{33} + 1$$

$$= (2 q_1^2 - 1 + 2 q_2^2) - (2 q_1^2 - 1 + 2 q_3^2) - (2 q_1^2 - 1 + 2 q_4^2) + 1$$

$$= -2 (-1 + q_1^2 - q_2^2 + q_3^2 + q_4^2) = -2 (-1 - 2 q_2^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2) = -2 (-1 - 2 q_2^2 + 1)$$

$$= 4 q_2^2$$

$$-r_{11} + r_{22} - r_{33} + 1$$

$$= -(2 q_1^2 - 1 + 2 q_2^2) + (2 q_1^2 - 1 + 2 q_3^2) - (2 q_1^2 - 1 + 2 q_4^2) + 1$$

$$= -2 (-1 + q_1^2 + q_2^2 - q_3^2 + q_4^2) = -2 (-1 - 2 q_3^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2) = -2 (-1 - 2 q_3^2 + 1)$$

$$= 4 q_3^2$$

$$-r_{11} - r_{22} + r_{33} + 1$$

$$= -(2 q_1^2 - 1 + 2 q_2^2) - (2 q_1^2 - 1 + 2 q_3^2) + (2 q_1^2 - 1 + 2 q_4^2) + 1$$

$$= -2 (-1 + q_1^2 + q_2^2 + q_3^2 - q_4^2) = -2 (-1 - 2 q_4^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2) = -2 (-1 - 2 q_4^2 + 1)$$

$$= 4 q_4^2$$

■ 3

$$q_1 = \frac{1}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1}$$

$$q_2 = -\frac{\text{Sign}[r_{32} - r_{23}]}{2} \sqrt{r_{11} - r_{22} - r_{33} + 1}$$

$$q_3 = -\frac{\text{Sign}[r_{13} - r_{31}]}{2} \sqrt{r_{22} - r_{33} - r_{11} + 1}$$

$$q_4 = -\frac{\text{Sign}[r_{21} - r_{12}]}{2} \sqrt{r_{33} - r_{11} - r_{22} + 1}$$

$$\left\{ \frac{1}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1}, -\frac{\text{Sign}[r_{32} - r_{23}]}{2} \sqrt{r_{11} - r_{22} - r_{33} + 1}, -\frac{\text{Sign}[r_{13} - r_{31}]}{2} \sqrt{r_{22} - r_{33} - r_{11} + 1}, -\frac{\text{Sign}[r_{21} - r_{12}]}{2} \sqrt{r_{33} - r_{11} - r_{22} + 1} \right\} // \text{InputForm}$$

{Sqrt[1 + r11 + r22 + r33]/2, -(Sqrt[1 + r11 - r22 - r33]*Sign[-r23 + r32])/2, -(Sqrt[1 - r11 + r22 - r33]*Sign[r13 - r31])/2, -(Sqrt[1 - r11 - r22 + r33]*Sign[-r12 + r21])/2}

Calibration

$${}^A_B R = R_x[\psi] \cdot R_y[\theta] \cdot R_z[\phi]$$

$$= \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$= \begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$${}^B_U = {}^A_B R \cdot {}^A_U$$

$${}^B_U = {}^A_B Q \otimes {}^A_U \otimes {}^A_B Q^*$$

$${}^C_U = {}^B_C R \cdot {}^B_U = {}^A_C R \cdot {}^A_U = {}^B_C R \cdot {}^A_B R \cdot {}^A_U;$$

$${}^C_U = {}^B_C Q \otimes {}^B_U \otimes {}^B_C Q^* = {}^B_C Q \otimes {}^A_B Q \otimes {}^A_U \otimes {}^A_B Q^* \otimes {}^B_C Q^*$$