



■ Axis/Angle → Rotation Matrix ${}^A_B R$

$$R[\gamma, r] = R_z[\alpha] \cdot R_y[\beta] \cdot R_z[\gamma] \cdot R_y[-\beta] \cdot R_z[-\alpha]$$

$$= \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[\gamma] & -\sin[\gamma] & 0 \\ \sin[\gamma] & \cos[\gamma] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[-\beta] & 0 & \sin[-\beta] \\ 0 & 1 & 0 \\ -\sin[-\beta] & 0 & \cos[-\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[-\alpha] & -\sin[-\alpha] & 0 \\ \sin[-\alpha] & \cos[-\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$${}^A_B R = \begin{pmatrix} r_x^2 (1 - \cos[\gamma]) + \cos[\gamma] & r_x r_y (1 - \cos[\gamma]) - r_z \sin[\gamma] & r_x r_z (1 - \cos[\gamma]) + r_y \sin[\gamma] \\ r_x r_y (1 - \cos[\gamma]) + r_z \sin[\gamma] & r_y^2 (1 - \cos[\gamma]) + \cos[\gamma] & r_y r_z (1 - \cos[\gamma]) - r_x \sin[\gamma] \\ r_x r_z (1 - \cos[\gamma]) - r_y \sin[\gamma] & r_y r_z (1 - \cos[\gamma]) + r_x \sin[\gamma] & r_z^2 (1 - \cos[\gamma]) + \cos[\gamma] \end{pmatrix}$$

■ Euler Angle (Roll $\phi \hat{x}_B$, Pitch $\theta \hat{y}_B$, Yaw $\psi \hat{z}_B$, ZYX) → Rotation Matrix ${}^A_B R$

$$R_{zyx} = R_z[\psi] R_y[\theta] R_x[\phi]$$

$$= \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix}$$

$${}^A_B R = \begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

■ Quaternion ${}^A_B \hat{q} \rightarrow$ Rotation Matrix ${}^A_B R$

$${}^A_B R = \begin{pmatrix} 2 q_1^2 - 1 + 2 q_2^2 & 2 (q_2 q_3 + q_1 q_4) & 2 (q_2 q_4 - q_1 q_3) \\ 2 (q_2 q_3 - q_1 q_4) & 2 q_1^2 - 1 + 2 q_3^2 & 2 (q_3 q_4 + q_1 q_2) \\ 2 (q_2 q_4 + q_1 q_3) & 2 (q_3 q_4 - q_1 q_2) & 2 q_1^2 - 1 + 2 q_4^2 \end{pmatrix}$$

$${}^A_P = {}^A_B R \cdot {}^B_P$$

$${}^A_B R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix}$$

■ Rotation Matrix ${}^A_B R \rightarrow$ Euler Angle (Roll $\phi \hat{x}_B$, Pitch $\theta \hat{y}_B$, Yaw $\psi \hat{z}_B$, ZYX)

$$\phi = \text{ArcTan}\left[\frac{r_{32}}{r_{33}}\right]$$

$$\theta = \text{ArcSin}[-r_{31}] = \text{ArcTan}\left[-\frac{r_{31}}{\sqrt{r_{32}^2 + r_{33}^2}}\right]$$

$$\psi = \text{ArcTan}\left[\frac{r_{21}}{r_{11}}\right]$$

■ Rotation Matrix ${}^A_B R \rightarrow \text{Axis/Angle}$

$$\gamma = \text{ArcCos} \left[\frac{r_{11} + r_{22} + r_{33} - 1}{2} \right]$$

$$\hat{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \frac{1}{2 \sin[\gamma]} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

$$r_x^2 + r_y^2 + r_z^2 = 1$$

■ Rotation Matrix ${}^A_B R \rightarrow \text{Quaternion } {}^A_B \hat{q}$

$${}^A_B \hat{q} = (q_1 \quad q_2 \quad q_3 \quad q_4) = \left(\cos\left[\frac{\gamma}{2}\right] \quad -r_x \sin\left[\frac{\gamma}{2}\right] \quad -r_y \sin\left[\frac{\gamma}{2}\right] \quad -r_z \sin\left[\frac{\gamma}{2}\right] \right)$$

$$q_1 = \frac{1}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1}$$

$$q_2 = -\frac{\text{Sign}[r_{32} - r_{23}]}{2} \sqrt{r_{11} - r_{22} - r_{33} + 1}$$

$$q_3 = -\frac{\text{Sign}[r_{13} - r_{31}]}{2} \sqrt{r_{22} - r_{33} - r_{11} + 1}$$

$$q_4 = -\frac{\text{Sign}[r_{21} - r_{12}]}{2} \sqrt{r_{33} - r_{11} - r_{22} + 1}$$

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

Quaternion

■ Quaternion conjugate

$${}^A_B \hat{q}^* = {}^B_A \hat{q} = (q_1 \quad -q_2 \quad -q_3 \quad -q_4)$$

■ Quaternion product $\otimes$

$${}^A_C \hat{q} = {}^B_C \hat{q} \otimes {}^A_B \hat{q}$$

$$\mathbf{a} \otimes \mathbf{b} = (a_1 \quad a_2 \quad a_3 \quad a_4) \otimes (b_1 \quad b_2 \quad b_3 \quad b_4)$$

$$= (a_1 \quad a_2 \quad a_3 \quad a_4) \cdot \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ -b_2 & b_1 & -b_4 & b_3 \\ -b_3 & b_4 & b_1 & -b_2 \\ -b_4 & -b_3 & b_2 & b_1 \end{pmatrix} = \begin{pmatrix} a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4 \\ a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3 \\ a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2 \\ a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1 \end{pmatrix}^T$$

■ Same vector from frame A to frame B

$${}^B_U = {}^A_B \hat{q} \otimes {}^A_U \otimes {}^A_B \hat{q}^*$$