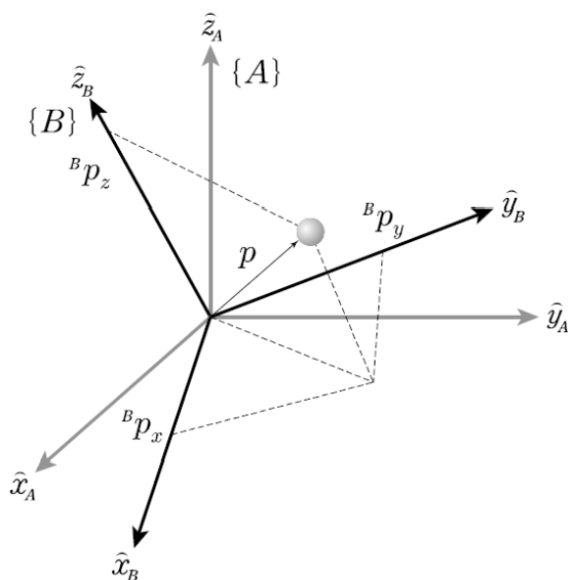


Position And Orientation



좌표계 {A}, {B} 에서 본 점 P의 위치

$$\text{Direction Vector : } \mathbf{P} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

■ 3D orthonormal basis → unit vectors

$$\hat{x}_A, \hat{y}_A, \hat{z}_A$$

$$\hat{\mathbf{A}} = \begin{pmatrix} \hat{x}_{Ax} & \hat{y}_{Ax} & \hat{z}_{Ax} \\ \hat{x}_{Ay} & \hat{y}_{Ay} & \hat{z}_{Ay} \\ \hat{x}_{Az} & \hat{y}_{Az} & \hat{z}_{Az} \end{pmatrix}$$

$$|\hat{x}_A| = |\hat{y}_A| = |\hat{z}_A| = 1$$

$$\hat{x}_A \cdot \hat{y}_A = 0, \hat{y}_A \cdot \hat{z}_A = 0, \hat{z}_A \cdot \hat{x}_A = 0$$

$$\hat{x}_A \times \hat{y}_A = \hat{z}_A$$

■ 상대위치표현 ${}^A\mathbf{P}$, ${}^B\mathbf{P}$ 의 표현

$$\begin{aligned} \mathbf{P} &= {}^A\mathbf{P} = {}^A p_x \hat{x}_A + {}^A p_y \hat{y}_A + {}^A p_z \hat{z}_A \\ &= {}^B\mathbf{P} = {}^B p_x \hat{x}_B + {}^B p_y \hat{y}_B + {}^B p_z \hat{z}_B \end{aligned}$$

$$\begin{cases} {}^A p_x, {}^A p_y, {}^A p_z : \mathbf{P} \text{의 } \hat{x}_A, \hat{y}_A, \hat{z}_A \text{ 방향 성분} \\ {}^B p_x, {}^B p_y, {}^B p_z : \mathbf{P} \text{의 } \hat{x}_B, \hat{y}_B, \hat{z}_B \text{ 방향 성분} \end{cases}$$

■ ${}^A\mathbf{P} = {}^A\mathbf{R}_B \cdot {}^B\mathbf{P}$

{B} 좌표계 성분을 {A} 좌표계 성분으로 변환

$$\begin{aligned}
{}^A\mathbf{P} &= \begin{pmatrix} {}^A p_x \\ {}^A p_y \\ {}^A p_z \end{pmatrix} = \begin{pmatrix} {}^A\mathbf{P} \cdot \hat{x}_A \\ {}^A\mathbf{P} \cdot \hat{y}_A \\ {}^A\mathbf{P} \cdot \hat{z}_A \end{pmatrix} = \begin{pmatrix} {}^B\mathbf{P} \cdot \hat{x}_A \\ {}^B\mathbf{P} \cdot \hat{y}_A \\ {}^B\mathbf{P} \cdot \hat{z}_A \end{pmatrix} = \begin{pmatrix} ({}^B p_x \hat{x}_B + {}^B p_y \hat{y}_B + {}^B p_z \hat{z}_B) \cdot \hat{x}_A \\ ({}^B p_x \hat{x}_B + {}^B p_y \hat{y}_B + {}^B p_z \hat{z}_B) \cdot \hat{y}_A \\ ({}^B p_x \hat{x}_B + {}^B p_y \hat{y}_B + {}^B p_z \hat{z}_B) \cdot \hat{z}_A \end{pmatrix} \\
&= \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix} \cdot \begin{pmatrix} {}^B p_x \\ {}^B p_y \\ {}^B p_z \end{pmatrix} \\
&= {}^A_B\mathbf{R} \cdot {}^B\mathbf{P}
\end{aligned}$$

→ 좌표계 {B}에 대해 표현된 점P의 위치 ${}^B\mathbf{P}$ 가 좌표계 {A}의 좌표값 ${}^A\mathbf{P}$ 로 회전행렬 ${}^A_B\mathbf{R}$ 을 통하여 변환

■ Rotation Matrix ${}^A_B\mathbf{R}$

좌표계 {A}에 대한 좌표계 {B}의 회전행렬

$$\begin{aligned}
\hat{x}_B &= \hat{x}_B \cdot \hat{x}_A + \hat{y}_B \cdot \hat{x}_A + \hat{z}_B \cdot \hat{x}_A \\
\hat{y}_B &= \hat{x}_B \cdot \hat{y}_A + \hat{y}_B \cdot \hat{y}_A + \hat{z}_B \cdot \hat{y}_A \\
\hat{z}_B &= \hat{x}_B \cdot \hat{z}_A + \hat{y}_B \cdot \hat{z}_A + \hat{z}_B \cdot \hat{z}_A
\end{aligned}$$

$${}^A_B\mathbf{R} = \begin{pmatrix} \hat{x}_B & \hat{y}_B & \hat{z}_B \end{pmatrix} = \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix}$$

$\hat{x}_B \ \hat{y}_B \ \hat{z}_B$ 를 각각 {A} 좌표성분 $(\hat{x}_A \ \hat{y}_A \ \hat{z}_A)$ 으로 표현한 식들의 조합으로 {A}에서 본 {B}의 표현식

좌표계 {A}와 {B}사이의 변환 행렬으로 {B}의 {A}에 대한 상대관계 또는 상대 회전량

물체의 방위 (상대적 회전량)를 나타냄

■ 회전행렬의 성질 (${}^B_A\mathbf{R} = {}^A_B\mathbf{R}^{-1} = {}^A_B\mathbf{R}^T$)

좌표계 {B}에 대한 좌표계 {A}의 회전행렬 : ${}^B_A\mathbf{R}$

좌표계 {A}에 대한 좌표계 {B}의 회전행렬 : ${}^A_B\mathbf{R}$

$${}^B_A\mathbf{R} = \begin{pmatrix} \hat{x}_A \cdot \hat{x}_B & \hat{y}_A \cdot \hat{x}_B & \hat{z}_A \cdot \hat{x}_B \\ \hat{x}_A \cdot \hat{y}_B & \hat{y}_A \cdot \hat{y}_B & \hat{z}_A \cdot \hat{y}_B \\ \hat{x}_A \cdot \hat{z}_B & \hat{y}_A \cdot \hat{z}_B & \hat{z}_A \cdot \hat{z}_B \end{pmatrix} \Leftrightarrow {}^A_B\mathbf{R} = \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix}$$

$${}^B_A\mathbf{R} = {}^A_B\mathbf{R}^T$$

$$\begin{cases} {}^A\mathbf{P} = {}^A_B\mathbf{R} \cdot {}^B\mathbf{P} \\ {}^B\mathbf{P} = {}^B_A\mathbf{R} \cdot {}^A\mathbf{P} \end{cases} \Rightarrow {}^A\mathbf{P} = {}^A_B\mathbf{R} \cdot {}^B\mathbf{P} = {}^A_B\mathbf{R} \cdot {}^B_A\mathbf{R} \cdot {}^A\mathbf{P} \Rightarrow {}^A_B\mathbf{R} \cdot {}^B_A\mathbf{R} = \mathbf{I}$$

$${}^B_A\mathbf{R} = {}^A_B\mathbf{R}^{-1} = {}^A_B\mathbf{R}^T$$

■ Cartesian coordinate (직교좌표계)

$$|\hat{x}_A| = |\hat{x}_B| = 1$$

■ Inner Product Space

$$\hat{x}_B \cdot \hat{x}_A = |\hat{x}_B| |\hat{x}_A| \cos[\gamma]$$

■ Law of Cosines

$$|\overrightarrow{\hat{x}_A \hat{x}_B}|^2 = |\hat{x}_A|^2 + |\hat{x}_B|^2 - 2 |\hat{x}_A| |\hat{x}_B| \cos[\gamma]$$

$$\cos[\gamma] = 1 - \frac{(\hat{x}_{B_x} - \hat{x}_{A_x})^2 + (\hat{x}_{B_y} - \hat{x}_{A_y})^2 + (\hat{x}_{B_z} - \hat{x}_{A_z})^2}{2}$$

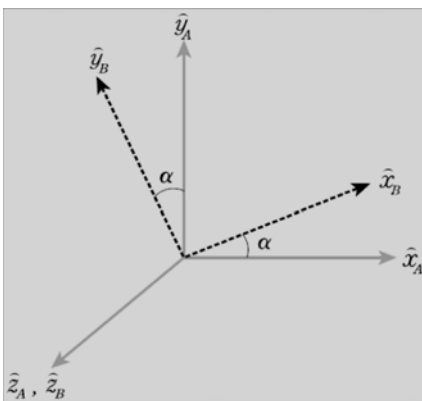
$$\hat{x}_B \cdot \hat{x}_A = 1 - \frac{(\hat{x}_{B_x} - \hat{x}_{A_x})^2 + (\hat{x}_{B_y} - \hat{x}_{A_y})^2 + (\hat{x}_{B_z} - \hat{x}_{A_z})^2}{2}$$

■ ${}^A_B R$: Transformation Matrix (변환행렬)

$$\begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix} =$$

$$\begin{pmatrix} 1 - \frac{(\hat{x}_{B_x} - \hat{x}_{A_x})^2 + (\hat{x}_{B_y} - \hat{x}_{A_y})^2 + (\hat{x}_{B_z} - \hat{x}_{A_z})^2}{2} & 1 - \frac{(\hat{y}_{B_x} - \hat{x}_{A_x})^2 + (\hat{y}_{B_y} - \hat{x}_{A_y})^2 + (\hat{y}_{B_z} - \hat{x}_{A_z})^2}{2} & 1 - \frac{(\hat{z}_{B_x} - \hat{x}_{A_x})^2 + (\hat{z}_{B_y} - \hat{x}_{A_y})^2 + (\hat{z}_{B_z} - \hat{x}_{A_z})^2}{2} \\ 1 - \frac{(\hat{x}_{B_x} - \hat{y}_{A_x})^2 + (\hat{x}_{B_y} - \hat{y}_{A_y})^2 + (\hat{x}_{B_z} - \hat{y}_{A_z})^2}{2} & 1 - \frac{(\hat{y}_{B_x} - \hat{y}_{A_x})^2 + (\hat{y}_{B_y} - \hat{y}_{A_y})^2 + (\hat{y}_{B_z} - \hat{y}_{A_z})^2}{2} & 1 - \frac{(\hat{z}_{B_x} - \hat{y}_{A_x})^2 + (\hat{z}_{B_y} - \hat{y}_{A_y})^2 + (\hat{z}_{B_z} - \hat{y}_{A_z})^2}{2} \\ 1 - \frac{(\hat{x}_{B_x} - \hat{z}_{A_x})^2 + (\hat{x}_{B_y} - \hat{z}_{A_y})^2 + (\hat{x}_{B_z} - \hat{z}_{A_z})^2}{2} & 1 - \frac{(\hat{y}_{B_x} - \hat{z}_{A_x})^2 + (\hat{y}_{B_y} - \hat{z}_{A_y})^2 + (\hat{y}_{B_z} - \hat{z}_{A_z})^2}{2} & 1 - \frac{(\hat{z}_{B_x} - \hat{z}_{A_x})^2 + (\hat{z}_{B_y} - \hat{z}_{A_y})^2 + (\hat{z}_{B_z} - \hat{z}_{A_z})^2}{2} \end{pmatrix}$$

■ 방위의 표현: 회전행렬

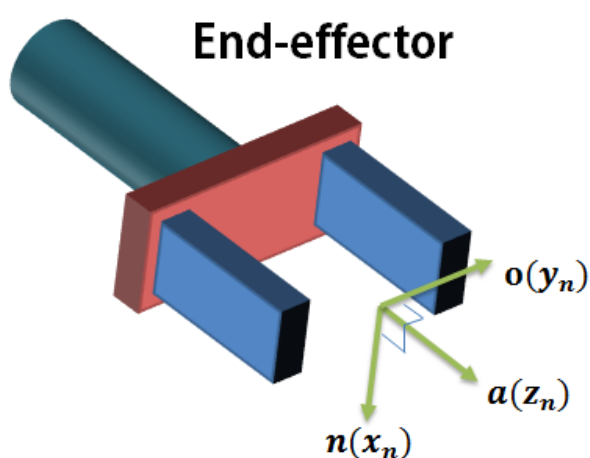


좌표계 {A} 가 $\hat{z}_A$ 축을 기준으로 α 만큼 회전한 좌표계를 {B} 라고 할 경우 회전행렬 ${}^A_B R$ 을 구하라.

$${}^A_B R = \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix} = \begin{pmatrix} \cos[\alpha] & \cos[90^\circ + \alpha] & \cos[90^\circ] \\ \cos[90^\circ - \alpha] & \cos[\alpha] & \cos[90^\circ] \\ \cos[90^\circ] & \cos[90^\circ] & \cos[0^\circ] \end{pmatrix} = \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Euler Angle (Roll ϕ $\hat{x}_B$, Pitch θ $\hat{y}_B$, Yaw ψ $\hat{z}_B$, ZYX)

- n(x): Normal, o(y): Orientation, a(z): Approach



- Unit Column Vector

$$\begin{aligned} \mathbf{n} &= (n_x \ n_y \ n_z)^T \\ \mathbf{o} &= (o_x \ o_y \ o_z)^T \\ \mathbf{a} &= (a_x \ a_y \ a_z)^T \end{aligned}$$

- Direction Cosine Matrix

$$R = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}$$

$$\mathbf{n} \cdot \mathbf{o} = \mathbf{o} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{n} = 0$$

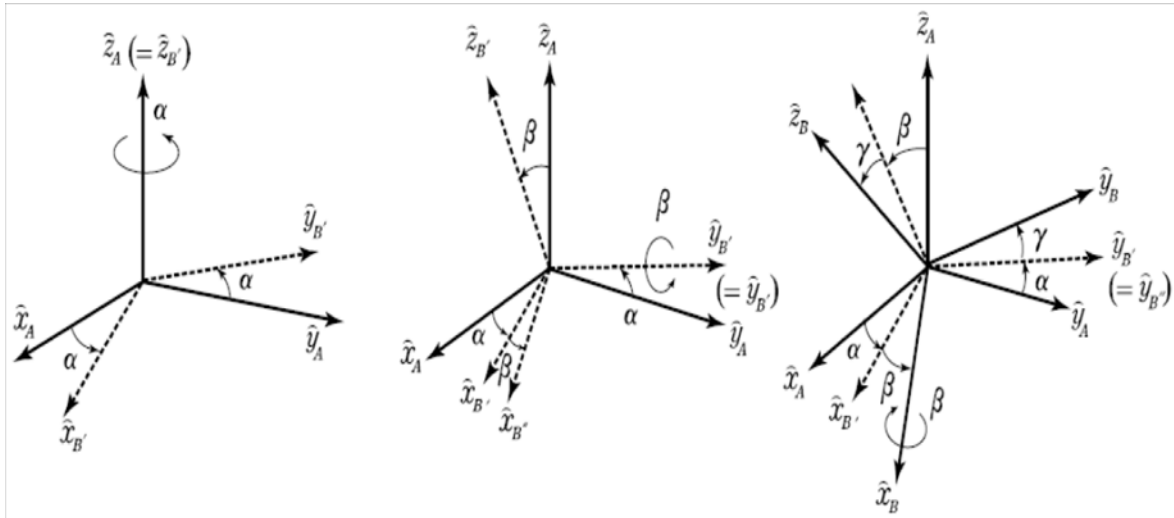
$$\mathbf{n} \times \mathbf{o} = \mathbf{a}$$

$$|\mathbf{n}| = |\mathbf{o}| = |\mathbf{a}| = 1$$

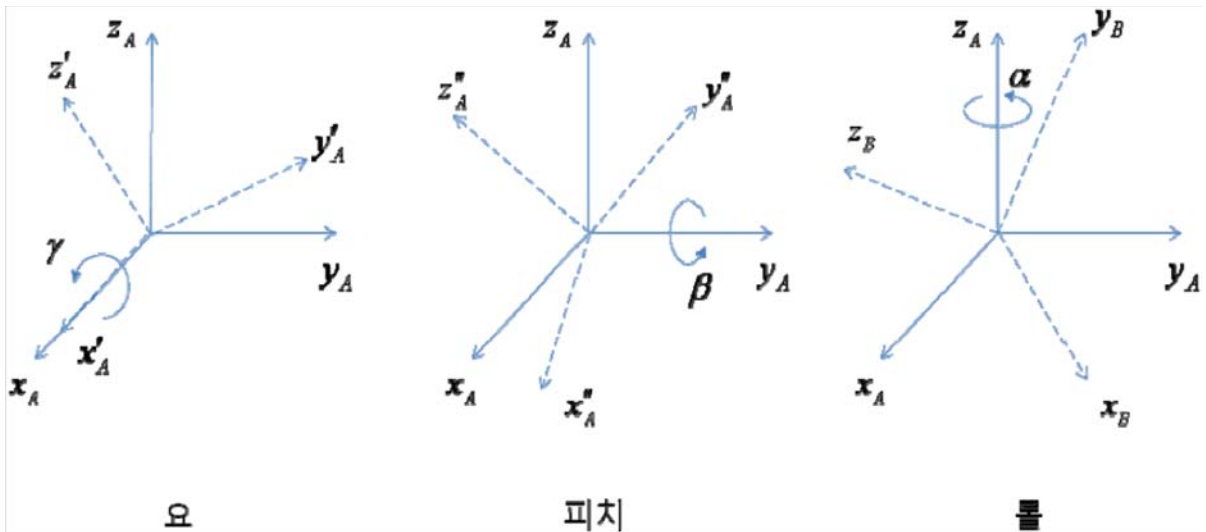
- Orthogonal Matrix

$$R^T \cdot R = R \cdot R^T = \mathbf{I}, \quad R^T = R^{-1}$$

■ Z-Y-X Euler Angle (이동 좌표계)



■ X-Y-Z (Roll-Pitch-Yaw) (고정 좌표계)



Euler Angle (Roll ϕ $\hat{x}_B$, Pitch θ $\hat{y}_B$, Yaw ψ $\hat{z}_B$, ZYX) $\leftrightarrow$ Rotation matrix ${}^A_B R$

■ Euler Angle (Roll ϕ $\hat{x}_B$, Pitch θ $\hat{y}_B$, Yaw ψ $\hat{z}_B$, ZYX) $\rightarrow$ Rotation Matrix ${}^A_B R$

X axis (Roll) $\phi \rightarrow R_x[\phi] \rightarrow \{-\pi, \pi\}$

Y axis (Pitch) $\theta \rightarrow R_y[\theta] \rightarrow \{-\pi/2, \pi/2\}$

Z axis (Yaw) $\psi \rightarrow R_z[\psi] \rightarrow \{-\pi, \pi\}$

$$R_{zyx} = R_z[\psi] R_y[\theta] R_x[\phi]$$

$$\begin{aligned}
&= \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \\
&= \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} \\
&= \begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}
\end{aligned}$$

■ Rotation Matrix ${}^A_B R \rightarrow$ Euler Angle (Roll $\phi \hat{x}_B$, Pitch $\theta \hat{y}_B$, Yaw $\psi \hat{z}_B$, ZYX)

■ $\theta = \{-\pi/2, \pi/2\}$

$$\phi \text{ (Roll)} = \text{ArcTan}\left[\frac{r_{32}}{r_{33}}\right]$$

$$\theta \text{ (Pitch)} = \text{ArcSin}[-r_{31}] = \text{ArcTan}\left[-\frac{r_{31}}{\sqrt{r_{32}^2 + r_{33}^2}}\right]$$

$$\psi \text{ (Yaw)} = \text{ArcTan}\left[\frac{r_{21}}{r_{11}}\right]$$

■ $\theta = \{\pi/2, 3\pi/2\}$

$$\phi = \text{ArcTan}\left[\frac{-r_{32}}{-r_{33}}\right]$$

$$\theta = \text{ArcSin}[-r_{31}]$$

$$\psi = \text{ArcTan}\left[\frac{-r_{21}}{-r_{11}}\right]$$

■ $\theta \text{ (Pitch)} = \pi/2, \cos[\theta] \approx 0$

$$r_{23} + r_{12} = (\sin[\theta] - 1) \sin[\phi + \psi]$$

$$r_{13} - r_{22} = (\sin[\theta] - 1) \cos[\phi + \psi]$$

$$r_{12} - r_{23} = (\sin[\theta] + 1) \sin[\phi - \psi]$$

$$r_{13} + r_{22} = (\sin[\theta] + 1) \cos[\phi - \psi]$$

$$\phi + \psi = \text{ArcTan}[r_{13} - r_{22}, r_{23} + r_{12}]$$

$$\theta = \text{ArcSin}[-r_{31}]$$

$$\phi - \psi = \text{ArcTan}[r_{13} + r_{22}, r_{12} - r_{23}]$$

Rotation matrix ${}^A_B\mathbf{R} \leftrightarrow \text{Euler Axis } \hat{\mathbf{r}} / \text{Angle } \gamma$

■ Angle γ and Axis $\mathbf{r}$

$$\mathbf{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}$$

$$|\mathbf{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2} = 1$$

$$\text{Sin}[\alpha] = \frac{r_y}{\sqrt{r_x^2 + r_y^2}}, \quad \text{Cos}[\alpha] = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\text{Sin}[\beta] = \frac{\sqrt{r_x^2 + r_y^2}}{\sqrt{r_x^2 + r_y^2 + r_z^2}}, \quad \text{Cos}[\beta] = \frac{r_z}{\sqrt{r_x^2 + r_y^2 + r_z^2}}$$

$$\alpha = \text{ArcTan}\left[\frac{r_y}{r_x}\right]$$

$$\beta = \text{ArcTan}\left[\frac{\sqrt{r_x^2 + r_y^2}}{r_z}\right]$$

■ Axis/Angle $\rightarrow$ Rotation Matrix ${}^A_B\mathbf{R}$

$$\mathbf{R}[\gamma, \mathbf{r}] = \mathbf{R}_z[\alpha] \cdot \mathbf{R}_y[\beta] \cdot \mathbf{R}_z[\gamma] \cdot \mathbf{R}_y[-\beta] \cdot \mathbf{R}_z[-\alpha]$$

$$= \begin{pmatrix} \text{Cos}[\alpha] & -\text{Sin}[\alpha] & 0 \\ \text{Sin}[\alpha] & \text{Cos}[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}[\beta] & 0 & \text{Sin}[\beta] \\ 0 & 1 & 0 \\ -\text{Sin}[\beta] & 0 & \text{Cos}[\beta] \end{pmatrix}.$$

$$\begin{pmatrix} \text{Cos}[\gamma] & -\text{Sin}[\gamma] & 0 \\ \text{Sin}[\gamma] & \text{Cos}[\gamma] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}[-\beta] & 0 & \text{Sin}[-\beta] \\ 0 & 1 & 0 \\ -\text{Sin}[-\beta] & 0 & \text{Cos}[-\beta] \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}[-\alpha] & -\text{Sin}[-\alpha] & 0 \\ \text{Sin}[-\alpha] & \text{Cos}[-\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

=

$$\begin{pmatrix} r_x^2 (1 - \text{Cos}[\gamma]) + \text{Cos}[\gamma] & r_x r_y (1 - \text{Cos}[\gamma]) - r_z \text{Sin}[\gamma] & r_x r_z (1 - \text{Cos}[\gamma]) + r_y \text{Sin}[\gamma] \\ r_x r_y (1 - \text{Cos}[\gamma]) + r_z \text{Sin}[\gamma] & r_y^2 (1 - \text{Cos}[\gamma]) + \text{Cos}[\gamma] & r_y r_z (1 - \text{Cos}[\gamma]) - r_x \text{Sin}[\gamma] \\ r_x r_z (1 - \text{Cos}[\gamma]) - r_y \text{Sin}[\gamma] & r_y r_z (1 - \text{Cos}[\gamma]) + r_x \text{Sin}[\gamma] & r_z^2 (1 - \text{Cos}[\gamma]) + \text{Cos}[\gamma] \end{pmatrix}$$

$$= \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

■ Rotation Matrix ${}^A_B\mathbf{R} \rightarrow \text{Axis/Angle}$

$$\gamma = \text{ArcCos}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right]$$

$$\hat{\mathbf{r}} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \frac{1}{2 \sin[\gamma]} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

$$r_x^2 + r_y^2 + r_z^2 = 1$$

- Axis/Angle → Rotation Matrix ${}^A_B \mathbf{R} \rightarrow$ Euler Angle (Roll $\phi \hat{x}_B$, Pitch $\theta \hat{y}_B$, Yaw $\psi \hat{z}_B$, ZYX)

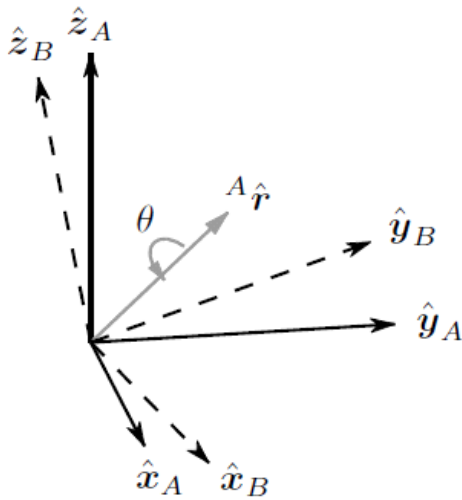
$$\phi \text{ (Roll)} = \text{ArcTan} \left[\frac{-\sin[\gamma] r_x + (-1 + \cos[\gamma]) r_y r_z}{-\cos[\gamma] + (-1 + \cos[\gamma]) r_z^2} \right]$$

$$\theta \text{ (Pitch)} = \text{ArcSin}[\sin[\gamma] r_y + (-1 + \cos[\gamma]) r_x r_z] = \text{ArcTan} \left[-\frac{r_{31}}{\sqrt{r_{32}^2 + r_{33}^2}} \right]$$

$$\psi \text{ (Yaw)} = \text{ArcTan} \left[\frac{-(-1 + \cos[\gamma]) r_x r_y + \sin[\gamma] r_z}{\cos[\gamma] - (-1 + \cos[\gamma]) r_x^2} \right]$$

Axis/Angle ↔ Rotation Matrix ${}^A_B \mathbf{R} \leftrightarrow$ Quaternion ${}^A_B \hat{\mathbf{q}} \leftrightarrow$ Euler Angle (Roll $\phi \hat{x}_B$, Pitch $\theta \hat{y}_B$, Yaw $\psi \hat{z}_B$, ZYX)

- Axis/Angle



- A rotation of angle θ around an axis ${}^A \hat{\mathbf{r}}$ defined in frame A

$${}^A \hat{\mathbf{r}} = (r_x \ r_y \ r_z)$$

■ Axis/Angle → Quaternion ${}^A_B\hat{q}$

■ An arbitrary orientation of frame B relative to frame A

$${}^A_B\hat{q} = (q_1 \ q_2 \ q_3 \ q_4) = \left(\cos\left[\frac{\theta}{2}\right] \ -r_x \sin\left[\frac{\theta}{2}\right] \ -r_y \sin\left[\frac{\theta}{2}\right] \ -r_z \sin\left[\frac{\theta}{2}\right] \right)$$

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

■ Quaternion ${}^A_B\hat{q} \rightarrow$ Rotation Matrix ${}^A_BR$

$${}^A_BR = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} 2q_1^2 - 1 + 2q_2^2 & 2(q_2q_3 + q_1q_4) & 2(q_2q_4 - q_1q_3) \\ 2(q_2q_3 - q_1q_4) & 2q_1^2 - 1 + 2q_3^2 & 2(q_3q_4 + q_1q_2) \\ 2(q_2q_4 + q_1q_3) & 2(q_3q_4 - q_1q_2) & 2q_1^2 - 1 + 2q_4^2 \end{pmatrix}$$

$$= \begin{pmatrix} r_x^2(1 - \cos[\theta]) + \cos[\theta] & r_x r_y(1 - \cos[\theta]) - r_z \sin[\theta] & r_x r_z(1 - \cos[\theta]) + r_y \sin[\theta] \\ r_x r_y(1 - \cos[\theta]) + r_z \sin[\theta] & r_y^2(1 - \cos[\theta]) + \cos[\theta] & r_y r_z(1 - \cos[\theta]) - r_x \sin[\theta] \\ r_x r_z(1 - \cos[\theta]) - r_y \sin[\theta] & r_y r_z(1 - \cos[\theta]) + r_x \sin[\theta] & r_z^2(1 - \cos[\theta]) + \cos[\theta] \end{pmatrix}$$

■ Half-Angle Identities

$$\cos\left[\frac{\theta}{2}\right] = \sqrt{\frac{1 + \cos[\theta]}{2}}$$

$$\sin\left[\frac{\theta}{2}\right] = \sqrt{\frac{1 - \cos[\theta]}{2}}$$

■ Rotation Matrix ${}^A_BR \rightarrow$ Quaternion ${}^A_B\hat{q}$

$$q_1 = \frac{1}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1}$$

$$q_2 = -\frac{\text{Sign}[r_{32} - r_{23}]}{2} \sqrt{r_{11} - r_{22} - r_{33} + 1}$$

$$q_3 = -\frac{\text{Sign}[r_{13} - r_{31}]}{2} \sqrt{r_{22} - r_{33} - r_{11} + 1}$$

$$q_4 = -\frac{\text{Sign}[r_{21} - r_{12}]}{2} \sqrt{r_{33} - r_{11} - r_{22} + 1}$$

■ Quaternion ${}^A_B\hat{q} \rightarrow$ Euler Angle (Roll $\phi \hat{x}_B$, Pitch $\theta \hat{y}_B$, Yaw $\psi \hat{z}_B$, ZYX)

$$\phi = \text{ArcTan}\left[\frac{{}^A_BR_{32}}{{}^A_BR_{33}}\right] = \text{ArcTan}\left[\frac{2(q_3q_4 - q_1q_2)}{2q_1^2 - 1 + 2q_4^2}\right]$$

$$\theta = -\text{ArcTan}\left[\frac{{}^A_BR_{31}}{\sqrt{1 - {}^A_BR_{31}^2}}\right] = -\text{ArcSin}[2(q_2q_4 + q_1q_3)]$$

$$\psi = \text{ArcTan}\left[\frac{{}^A_B R_{21}}{{}^A_B R_{11}}\right] = \text{ArcTan}\left[\frac{2(q_2 q_3 - q_1 q_4)}{2q_1^2 - 1 + 2q_2^2}\right]$$

■ Euler Angle (Roll $\phi \hat{x}_B$, Pitch $\theta \hat{y}_B$, Yaw $\psi \hat{z}_B$, ZYX) $\rightarrow$ Quaternion ${}^A_B \hat{q}$

$$R_{zyx} = R_z[\psi] R_y[\theta] R_x[\phi]$$

$$= \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix}$$

$$= \begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$q_1 = \frac{1}{2} \sqrt{r_{11} + r_{22} + r_{33} + 1}$$

$$= \frac{1}{2} \sqrt{1 + \cos[\theta] \cos[\phi] + \cos[\theta] \cos[\psi] + \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi]}$$

$$q_2 = -\frac{\text{Sign}[r_{32} - r_{23}]}{2} \sqrt{r_{11} - r_{22} - r_{33} + 1}$$

$$= -\frac{1}{2} \text{Sign}[\cos[\theta] \sin[\phi] + \cos[\psi] \sin[\phi] - \cos[\phi] \sin[\theta] \sin[\psi]]$$

$$\sqrt{1 - \cos[\theta] \cos[\phi] + \cos[\theta] \cos[\psi] - \cos[\phi] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi]}$$

$$q_3 = -\frac{\text{Sign}[r_{13} - r_{31}]}{2} \sqrt{r_{22} - r_{33} - r_{11} + 1}$$

$$= -\frac{1}{2} \text{Sign}[\sin[\theta] + \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi]]$$

$$\sqrt{1 - \cos[\theta] \cos[\phi] - \cos[\theta] \cos[\psi] + \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi]}$$

$$q_4 = -\frac{\text{Sign}[r_{21} - r_{12}]}{2} \sqrt{r_{33} - r_{11} - r_{22} + 1}$$

$$= -\frac{1}{2} \text{Sign}[-\cos[\psi] \sin[\theta] \sin[\phi] + \cos[\theta] \sin[\psi] + \cos[\phi] \sin[\psi]]$$

$$\sqrt{1 + \cos[\theta] \cos[\phi] - \cos[\theta] \cos[\psi] - \cos[\phi] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi]}$$

■ Quaternion ${}^A_B \hat{q} \rightarrow$ Axis/Angle

$${}^A_B R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} 2q_1^2 - 1 + 2q_2^2 & 2(q_2 q_3 + q_1 q_4) & 2(q_2 q_4 - q_1 q_3) \\ 2(q_2 q_3 - q_1 q_4) & 2q_1^2 - 1 + 2q_3^2 & 2(q_3 q_4 + q_1 q_2) \\ 2(q_2 q_4 + q_1 q_3) & 2(q_3 q_4 - q_1 q_2) & 2q_1^2 - 1 + 2q_4^2 \end{pmatrix}$$

$$\theta = \text{ArcCos}\left[\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right] = \text{ArcCos}[-2 + 3q_1^2 + q_2^2 + q_3^2 + q_4^2] = \text{ArcCos}[2q_1^2 - 1]$$

$$\hat{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} =$$

$$\frac{1}{2 \sin[\theta]} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix} = \frac{1}{2 \sin[\theta]} \begin{pmatrix} 2(q_3 q_4 - q_1 q_2) - 2(q_3 q_4 + q_1 q_2) \\ 2(q_2 q_4 - q_1 q_3) - 2(q_2 q_4 + q_1 q_3) \\ 2(q_2 q_3 - q_1 q_4) - 2(q_2 q_3 + q_1 q_4) \end{pmatrix} = \frac{1}{\sin[\theta]} \begin{pmatrix} -2q_1 q_2 \\ -2q_1 q_3 \\ -2q_1 q_4 \end{pmatrix}$$

$$q_1 = \cos\left[\frac{\theta}{2}\right] = \sqrt{\frac{1 + \cos[\theta]}{2}}$$

$$\sin\left[\frac{\theta}{2}\right] = \sqrt{\frac{1 - \cos[\theta]}{2}} = \sqrt{1 - q_1^2}$$

$${}^A_B \hat{q} = (q_1 \quad -r_x \sqrt{1 - q_1^2} \quad -r_y \sqrt{1 - q_1^2} \quad -r_z \sqrt{1 - q_1^2})$$

$$\theta = 2 \operatorname{ArcCos}[q_1] = \operatorname{ArcCos}[2 q_1^2 - 1]$$

$$\hat{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \frac{-1}{\sqrt{1 - q_1^2}} \begin{pmatrix} q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

Quaternion

■ Quaternion conjugate

$${}^A_B \hat{q}^* = {}^B_A \hat{q} = (q_1 \quad -q_2 \quad -q_3 \quad -q_4)$$

■ Quaternion product $\otimes$

$${}^A_C \hat{q} = {}^B_C \hat{q} \otimes {}^A_B \hat{q}$$

■ Hamilton rule

$$\mathbf{a} \otimes \mathbf{b} = (a_1 \quad a_2 \quad a_3 \quad a_4) \otimes (b_1 \quad b_2 \quad b_3 \quad b_4)$$

$$= (a_1 \quad a_2 \quad a_3 \quad a_4) \cdot \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ -b_2 & b_1 & -b_4 & b_3 \\ -b_3 & b_4 & b_1 & -b_2 \\ -b_4 & -b_3 & b_2 & b_1 \end{pmatrix}$$

$$= \{a_1 b_1 - \{a_2, a_3, a_4\} \cdot \{b_2, b_3, b_4\},$$

$$a_1 \{b_2, b_3, b_4\} + b_1 \{a_2, a_3, a_4\} + \{a_2, a_3, a_4\} \times \{b_2, b_3, b_4\}\}$$

$$(ab_1 \quad ab_2 \quad ab_3 \quad ab_4) = \begin{pmatrix} a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4 \\ a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3 \\ a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2 \\ a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1 \end{pmatrix}_T$$

■ 3D Vector

$${}^B_U = {}^A_B \hat{q} \otimes {}^A_U \otimes {}^A_B \hat{q}^*$$

$$(\theta \quad {}^B_{U_x} \quad {}^B_{U_y} \quad {}^B_{U_z}) = (q_1 \quad q_2 \quad q_3 \quad q_4) \otimes (\theta \quad {}^A_{U_x} \quad {}^A_{U_y} \quad {}^A_{U_z}) \otimes (q_1 \quad -q_2 \quad -q_3 \quad -q_4)$$

$$= (q_2 b_2 - q_3 b_3 - q_4 b_4 \quad q_1 b_2 + q_3 b_4 - q_4 b_3 \quad q_1 b_3 - q_2 b_4 + q_4 b_2 \quad q_1 b_4 + q_2 b_3 - q_3 b_2) \otimes (q_1 \quad -q_2 \quad -q_3 \quad -q_4)$$

$$\begin{aligned}
&= \{ (q_2 b_2 - q_3 b_3 - q_4 b_4) q_1 + (q_1 b_2 + q_3 b_4 - q_4 b_3) q_2 + \\
&\quad (q_1 b_3 - q_2 b_4 + q_4 b_2) q_3 + (q_1 b_4 + q_2 b_3 - q_3 b_2) q_4, \\
&\quad - (q_2 b_2 - q_3 b_3 - q_4 b_4) q_2 + (q_1 b_2 + q_3 b_4 - q_4 b_3) q_1 - \\
&\quad (q_1 b_3 - q_2 b_4 + q_4 b_2) q_4 + (q_1 b_4 + q_2 b_3 - q_3 b_2) q_3, \\
&\quad - (q_2 b_2 - q_3 b_3 - q_4 b_4) q_3 + (q_1 b_2 + q_3 b_4 - q_4 b_3) q_4 + \\
&\quad (q_1 b_3 - q_2 b_4 + q_4 b_2) q_1 - (q_1 b_4 + q_2 b_3 - q_3 b_2) q_2, \\
&\quad - (q_2 b_2 - q_3 b_3 - q_4 b_4) q_4 - (q_1 b_2 + q_3 b_4 - q_4 b_3) q_3 + \\
&\quad (q_1 b_3 - q_2 b_4 + q_4 b_2) q_2 + (q_1 b_4 + q_2 b_3 - q_3 b_2) q_1 \} \\
&= \{ 2 b_2 q_1 q_2, b_2 (q_1^2 - q_2^2 - q_3^2 - q_4^2) + 2 (b_3 (q_2 q_3 - q_1 q_4) + b_4 (q_1 q_3 + q_2 q_4)) , \\
&\quad b_3 (q_1^2 - q_2^2 + q_3^2 - q_4^2) + 2 (b_2 q_1 q_4 + b_4 (-q_1 q_2 + q_3 q_4)) , \\
&\quad b_4 (q_1^2 - q_2^2 - q_3^2 + q_4^2) + 2 (-b_2 q_1 q_3 + b_3 (q_1 q_2 + q_3 q_4)) \}
\end{aligned}$$

■ Rotation Matrix → Quaternion

$$\frac{1}{3} \begin{pmatrix} R_{11} + R_{22} + R_{33} & R_{23} - R_{32} & R_{31} - R_{13} & R_{12} - R_{21} \\ R_{23} - R_{32} & R_{11} - R_{22} - R_{33} & R_{21} + R_{12} & R_{31} + R_{13} \\ R_{31} - R_{13} & R_{21} + R_{12} & R_{22} - R_{11} - R_{33} & R_{32} + R_{23} \\ R_{12} - R_{21} & R_{31} + R_{13} & R_{32} + R_{23} & R_{33} - R_{11} - R_{22} \end{pmatrix}$$

Orientation from angular rate

■ A tri-axis gyroscope

■ The angular rate

$${}^S\omega = (0 \quad \omega_x \quad \omega_y \quad \omega_z)$$

■ The quaternion derivative

$${}^E\dot{q} = \frac{1}{2} {}^E\hat{q} \otimes {}^S\omega$$

■ The quaternion derivative measured at time t

$${}^E\dot{q}_{\omega, t} = \frac{1}{2} {}^E\hat{q}_{\text{est}, t-1} \otimes {}^S\omega_t$$

■ The orientation of the earth frame at time t

$${}^E\mathbf{q}_{\omega, t} = {}^E\mathbf{q}_{\text{est}, t-1} + {}^E\dot{q}_{\omega, t} \Delta t$$

The sampling period Δt

Orientation from vector observations

■ A tri-axis accelerometer (Linear accelerations due to motion)

the field of gravity

■ A tri-axis magnetometer (Local magnetic flux and distortions)

the earth ' s magnetic field

■ A quaternion

$${}^S\hat{q} = (q_1 \quad q_2 \quad q_3 \quad q_4)$$

■ A predefined reference direction

$${}^E\hat{d} = (0 \quad d_x \quad d_y \quad d_z)$$

■ The measured direction

$${}^S\hat{s} = (0 \quad s_x \quad s_y \quad s_z)$$

■ The objective function

$$f[{}^S\hat{q}, {}^E\hat{d}, {}^S\hat{s}] = {}^S\hat{q}^* \otimes {}^E\hat{d} \otimes {}^S\hat{q} - {}^S\hat{s}$$

■ A quaternion may be found

$$\min_{{}^S\hat{q} \in \mathbb{R}^4} f[{}^S\hat{q}, {}^E\hat{d}, {}^S\hat{s}]$$

The gradient descent algorithm

■ An orientation estimation of ${}^S\hat{q}_{n+1}$

$${}^S\hat{q}_{k+1} = {}^S\hat{q}_k - \mu \frac{\nabla f[{}^S\hat{q}, {}^E\hat{d}, {}^S\hat{s}]}{|\nabla f[{}^S\hat{q}, {}^E\hat{d}, {}^S\hat{s}]|}, \quad k = 0, 1, \dots, n$$

A step-size μ

■ The gradient of the solution surface (General form)

$$\nabla f[\hat{\mathbf{q}}_k, {}^E\hat{\mathbf{d}}, {}^S\hat{\mathbf{s}}] = \mathbf{J}^T[\hat{\mathbf{q}}_k, {}^E\hat{\mathbf{d}}] f[\hat{\mathbf{q}}_k, {}^E\hat{\mathbf{d}}, {}^S\hat{\mathbf{s}}]$$

■ A objective function

$$f[\hat{\mathbf{q}}_k, {}^E\hat{\mathbf{d}}, {}^S\hat{\mathbf{s}}] = \begin{pmatrix} 2 d_x (\frac{1}{2} - q_3^2 - q_4^2) + 2 d_y (q_1 q_4 + q_2 q_3) + 2 d_z (q_2 q_4 - q_1 q_3) - s_x \\ 2 d_x (q_2 q_3 - q_1 q_4) + 2 d_y (\frac{1}{2} - q_2^2 - q_4^2) + 2 d_z (q_1 q_2 + q_3 q_4) - s_y \\ 2 d_x (q_1 q_3 + q_2 q_4) + 2 d_y (q_3 q_4 - q_1 q_2) + 2 d_z (\frac{1}{2} - q_2^2 - q_3^2) - s_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}[\hat{\mathbf{q}}_k, {}^E\hat{\mathbf{d}}] = \begin{pmatrix} 2 d_y q_4 - 2 d_z q_3 & 2 d_y q_3 + 2 d_z q_4 & -4 d_x q_3 + 2 d_y q_2 - 2 d_z q_1 & -4 d_x q_4 + 2 d_y q_1 + 2 d_z q_2 \\ -2 d_x q_4 + 2 d_z q_2 & 2 d_x q_3 - 4 d_y q_2 + 2 d_z q_1 & 2 d_x q_2 + 2 d_z q_4 & -2 d_x q_1 - 4 d_y q_4 + 2 d_z q_3 \\ 2 d_x q_3 - 2 d_y q_2 & 2 d_x q_4 - 2 d_y q_1 - 4 d_z q_2 & 2 d_x q_1 + 2 d_y q_4 - 4 d_z q_3 & 2 d_x q_2 + 2 d_y q_3 \end{pmatrix}$$

The direction of gravity (the vertical axis, z axis)

■ An appropriate convention (the equations simplify)

$${}^E\hat{\mathbf{g}} = (0 \ 0 \ 0 \ 1)$$

■ Normalized accelerometer measurement

$${}^S\hat{\mathbf{a}} = (0 \ a_x \ a_y \ a_z)$$

■ A objective function

$$f_g[\hat{\mathbf{q}}, {}^S\hat{\mathbf{a}}] = \begin{pmatrix} 2 (q_2 q_4 - q_1 q_3) - a_x \\ 2 (q_1 q_2 + q_3 q_4) - a_y \\ 2 (\frac{1}{2} - q_2^2 - q_3^2) - a_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}_g[\hat{\mathbf{q}}] = \begin{pmatrix} -2 q_3 & 2 q_4 & -2 q_1 & 2 q_2 \\ 2 q_2 & 2 q_1 & 2 q_4 & 2 q_3 \\ 0 & -4 q_2 & -4 q_3 & 0 \end{pmatrix}$$

The earth's magnetic field (one horizontal axis, the vertical axis)

- An appropriate convention (the equations simplify)

$${}^E\hat{\mathbf{b}} = (0 \quad b_x \quad 0 \quad b_z)$$

- Normalized accelerometer measurement

$${}^S\hat{\mathbf{m}} = (0 \quad m_x \quad m_y \quad m_z)$$

- A objective function

$$\mathbf{f}_b[{}^S\hat{\mathbf{q}}, {}^E\hat{\mathbf{b}}, {}^S\hat{\mathbf{m}}] = \begin{pmatrix} 2 b_x (\frac{1}{2} - q_3^2 - q_4^2) + 2 b_z (q_2 q_4 - q_1 q_3) - m_x \\ 2 b_x (q_2 q_3 - q_1 q_4) + 2 b_z (q_1 q_2 + q_3 q_4) - m_y \\ 2 b_x (q_1 q_3 + q_2 q_4) + 2 b_z (\frac{1}{2} - q_2^2 - q_3^2) - m_z \end{pmatrix}$$

- Jacobian

$$\mathbf{J}_b[{}^S\hat{\mathbf{q}}, {}^E\hat{\mathbf{b}}] = \begin{pmatrix} -2 b_z q_3 & 2 b_z q_4 & -4 b_x q_3 - 2 b_z q_1 & -4 b_x q_4 + 2 b_z q_2 \\ -2 b_x q_4 + 2 b_z q_2 & 2 b_x q_3 + 2 b_z q_1 & 2 b_x q_2 + 2 b_z q_4 & -2 b_x q_1 + 2 b_z q_3 \\ 2 b_x q_3 & 2 b_x q_4 - 4 b_z q_2 & 2 b_x q_1 - 4 b_z q_3 & 2 b_x q_2 \end{pmatrix}$$

The solution surface → minimum

- The objective function

$$\mathbf{f}_{g,b}[{}^S\hat{\mathbf{q}}, {}^S\hat{\mathbf{a}}, {}^E\hat{\mathbf{b}}, {}^S\hat{\mathbf{m}}] = \begin{pmatrix} \mathbf{f}_g[{}^S\hat{\mathbf{q}}, {}^S\hat{\mathbf{a}}] \\ \mathbf{f}_b[{}^S\hat{\mathbf{q}}, {}^E\hat{\mathbf{b}}, {}^S\hat{\mathbf{m}}] \end{pmatrix}$$

- Jacobian

$$\mathbf{J}_{g,b}[{}^S\hat{\mathbf{q}}, {}^E\hat{\mathbf{b}}] = \begin{pmatrix} \mathbf{J}_g^T[{}^S\hat{\mathbf{q}}] \\ \mathbf{J}_b^T[{}^S\hat{\mathbf{q}}, {}^E\hat{\mathbf{b}}] \end{pmatrix}$$

The estimate orientation ${}^S\hat{\mathbf{q}}_{\nabla, t}$

$${}^S\hat{\mathbf{q}}_{\nabla, t} = {}^S\hat{\mathbf{q}}_{\text{est}, t} - \mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|}$$

- The convergence rate governed by $\mu_t \geq$ the physical rate

A previous estimate of orientation ${}^S\hat{\mathbf{q}}_{\text{est}, t-1}$

■ The objective function gradient

$$\nabla f = \begin{Bmatrix} \mathbf{J}_g^T [\hat{\mathbf{q}}_{\text{est}, t-1}] \mathbf{f}_g [\hat{\mathbf{q}}_{\text{est}, t-1}, \mathbf{s}\hat{\mathbf{a}}_t] \\ \mathbf{J}_{g,b}^T [\hat{\mathbf{q}}_{\text{est}, t-1}, \mathbf{E}\hat{\mathbf{b}}] \mathbf{f}_{g,b} [\hat{\mathbf{q}}_{\text{est}, t-1}, \mathbf{s}\hat{\mathbf{a}}, \mathbf{E}\hat{\mathbf{b}}, \mathbf{s}\hat{\mathbf{m}}] \end{Bmatrix}$$

The convergence rate of $\hat{\mathbf{q}}_{\nabla, t}$

The physical orientation rate $\hat{\mathbf{q}}_{\omega, t}$

■ An optimal value of the step-size

$$\mu_t = \alpha \|\hat{\mathbf{q}}_{\omega, t}\| \Delta t, \alpha > 1$$

Filter fusion algorithm

■ An estimated orientation

$$\hat{\mathbf{q}}_{\text{est}, t} = \gamma_t \hat{\mathbf{q}}_{\nabla, t} + (1 - \gamma_t) \hat{\mathbf{q}}_{\omega, t}, \quad 0 \leq \gamma_t \leq 1$$

■ $\beta, \frac{\mu_t}{\Delta t}$

$$\text{The weighted divergence of } \hat{\mathbf{q}}_{\omega} \stackrel{\beta}{=} \text{The weighted convergence of } \hat{\mathbf{q}}_{\nabla} \stackrel{\frac{\mu_t}{\Delta t}}{=}$$

■ The magnitude of a quaternion derivative == The gyroscope measurement error

$$(1 - \gamma_t) \beta = \gamma_t \frac{\mu_t}{\Delta t}$$

$$\gamma_t = \frac{\beta}{\frac{\mu_t}{\Delta t} + \beta}$$

■ The optimal fusion

■ the convergence rate of $\hat{\mathbf{q}}_{\nabla} \geq \alpha \rightarrow$ the physical rate of change of orientation

$$\hat{\mathbf{q}}_{\nabla, t} \approx -\mu_t \frac{\nabla f}{\|\nabla f\|}$$

$$\gamma_t \approx \frac{\beta \Delta t}{\mu_t}$$

Magnetic distortion compensation

■ Declination errors

The horizontal plane : the earth ' s surface (Heading)

■ Inclination errors

The vertical plane : the earth ' s surface (Sensor ' s attitude)

■ The measured direction of the earth's magnetic field in the earth frame at time t

$${}^E\hat{\mathbf{h}}_t = (0 \quad h_x \quad h_y \quad h_z) = {}^S\hat{\mathbf{q}}_{\text{est},t-1} \otimes {}^S\hat{\mathbf{m}}_t \otimes {}^S\hat{\mathbf{q}}_{\text{est},t-1}^*$$

$${}^E\hat{\mathbf{b}}_t = (0 \quad \sqrt{h_x^2 + h_y^2} \quad 0 \quad h_z)$$

Gyroscope bias drift compensation

■ With temperature and with motion

Karman - based approaches → estimate the gyroscope bias

Mahony et al → gyroscope bias drift (the integral feedback of the error)

The normalized direction of the estimated error in the rate of change of orientation ${}^S\dot{\hat{\mathbf{q}}}_\epsilon$

■ The gyroscope bias ${}^S\omega_b$

$${}^S\omega_{b,t} = \zeta \sum_t {}^S\omega_{\epsilon,t} \Delta t$$

The integral gain ζ

■ DC component of ${}^S\omega_\epsilon$

$${}^S\omega_{\epsilon,t} = 2 {}^S\hat{\mathbf{q}}_{\text{est},t-1}^* \otimes {}^S\dot{\hat{\mathbf{q}}}_{\epsilon,t}$$

■ Gyroscope measurements ${}^S\omega_c$

$${}^S\omega_{c,t} = {}^S\omega_t - {}^S\omega_{b,t}$$

Filter gains

The estimated mean zero gyroscope measurement error of each axis $\tilde{\omega}_\beta$

■ The filter gain

$$\beta = \left| \frac{1}{2} \mathbf{q} \otimes (\mathbf{0} \quad \tilde{\omega}_\beta \quad \tilde{\omega}_\beta \quad \tilde{\omega}_\beta) \right| = \sqrt{\frac{3}{4}} \tilde{\omega}_\beta$$

The estimated rate of gyroscope bias drift in each axis $\dot{\omega}_\xi$

■ The filter gain

$$\xi = \xi \sqrt{\frac{3}{4}} \tilde{\omega}_\xi$$

An efficient orientation filter for inertial and inertial/magnetic sensor arrays

■ An estimated orientation

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \frac{\beta \Delta t}{\mu_t} \left(-\mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) + (1 - \theta) (\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} \Delta t)$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = -\beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \Delta t + \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} \Delta t$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} \Delta t - \beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \Delta t$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \left(\hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} - \beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \left(\hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} - \beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \left(\hat{\mathbf{e}}_{\mathbf{q}_{\omega}, t} - \beta \hat{\mathbf{e}}_{\mathbf{e}, t} \right) \Delta t$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} \Delta t$$

$$\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t} = \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} + \left(\frac{1}{2} \hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}, t-1} \otimes \mathbf{s}_{\omega_t} - \beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

The estimated orientation rate $\hat{\mathbf{e}}_{\mathbf{q}_{\text{est}}}$

The rate of change of orientation measured by the gyroscopes $\hat{\mathbf{e}}_{\mathbf{q}_{\omega}}$

The direction of the estimated error $\hat{\mathbf{e}}_{\mathbf{e}}$

■ Simplified to equation

$$\hat{\mathbf{q}}_{\text{est},t} = \hat{\mathbf{q}}_{\text{est},t-1} + \dot{\hat{\mathbf{q}}}_{\text{est},t} \Delta t$$

■ The estimated rate of change of orientation

$$\dot{\hat{\mathbf{q}}}_{\text{est},t} = \dot{\mathbf{q}}_{\omega,t} - \beta \dot{\hat{\mathbf{q}}}_{\epsilon,t}$$

■ The quaternion derivative measured at time t

$$\dot{\mathbf{q}}_{\omega,t} = \frac{1}{2} \hat{\mathbf{q}}_{\text{est},t-1} \otimes \mathbf{s}_{\omega,t}$$

■ The direction of the error of $\hat{\mathbf{q}}_{\text{est},t}$

$$\hat{\mathbf{q}}_{\epsilon,t} = \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|}$$

■ The objective function

$$\mathbf{f}_{g,b}[\hat{\mathbf{q}}, \mathbf{s}_{\mathbf{a}}, \mathbf{e}_{\mathbf{b}}, \mathbf{s}_{\mathbf{m}}] = \begin{pmatrix} \mathbf{f}_g[\hat{\mathbf{q}}, \mathbf{s}_{\mathbf{a}}] \\ \mathbf{f}_b[\hat{\mathbf{q}}, \mathbf{e}_{\mathbf{b}}, \mathbf{s}_{\mathbf{m}}] \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}_{g,b}[\hat{\mathbf{q}}, \mathbf{e}_{\mathbf{b}}] = \begin{pmatrix} \mathbf{J}_g^T[\hat{\mathbf{q}}] \\ \mathbf{J}_b^T[\hat{\mathbf{q}}, \mathbf{e}_{\mathbf{b}}] \end{pmatrix}$$

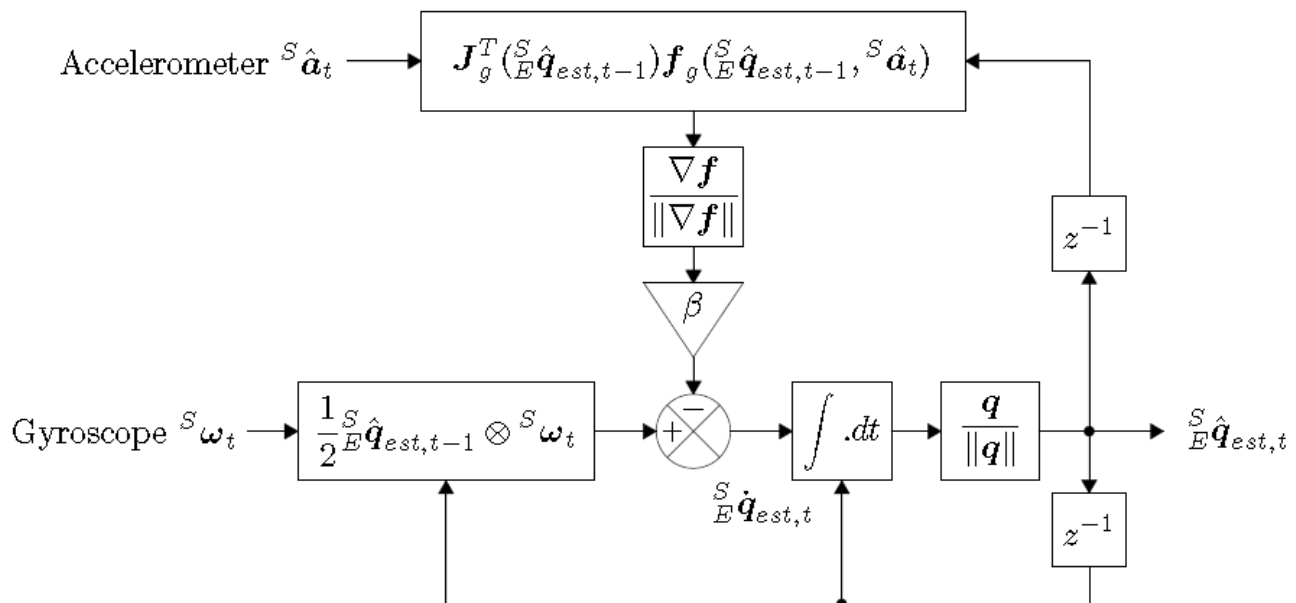
■ The gradient of the solution surface (General form)

$$\nabla \mathbf{f} = \mathbf{J}^T \mathbf{f}$$

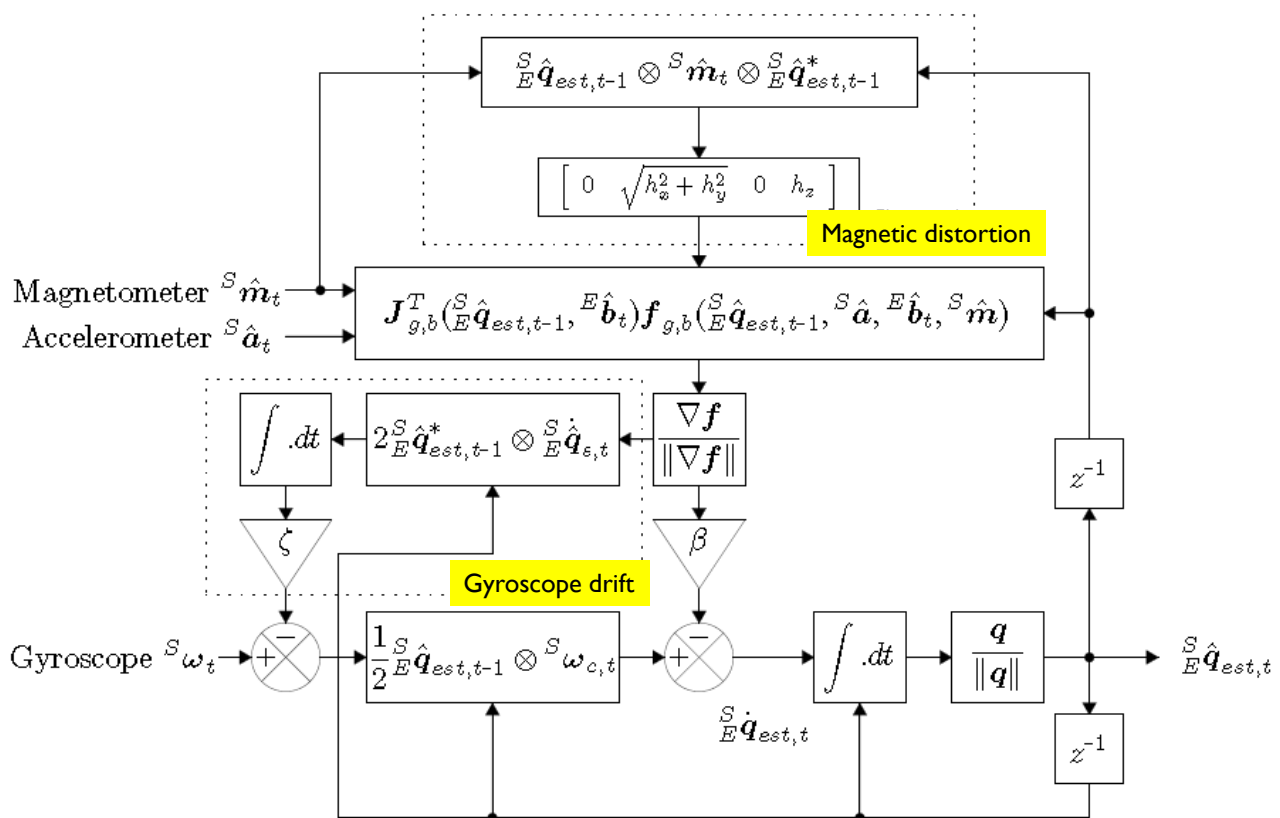
$$\nabla \mathbf{f}_{g,b} = \mathbf{J}_{g,b}[\hat{\mathbf{q}}, \mathbf{e}_{\mathbf{b}}]^T \mathbf{f}_{g,b}[\hat{\mathbf{q}}, \mathbf{s}_{\mathbf{a}}, \mathbf{e}_{\mathbf{b}}, \mathbf{s}_{\mathbf{m}}]$$

Block diagram

■ IMU



■ MARG



MATLAB

■ Update

```
obj = Update (obj, Gyroscope, Accelerometer, Magnetometer)
```

■ % short name local variable for readability

```
q = obj.Quaternion;
```

■ % Normalise accelerometer measurement

```
if (norm (Accelerometer) == 0), return; end % handle NaN
Accelerometer = Accelerometer / norm (Accelerometer); % normalise magnitude
```

■ % Normalise magnetometer measurement

```
if (norm (Magnetometer) == 0), return; end % handle NaN
Magnetometer = Magnetometer / norm (Magnetometer); % normalise magnitude
```

■ % Reference direction of Earth's magnetic feild

```
h = quaternProd (q, quaternProd ([0 Magnetometer], quaternConj (q)));
b = [0 norm ([h (2) h (3)]) 0 h (4)];
```

■ % Gradient decent algorithm corrective step

```
F =
(
2 * (q (2) * q (4) - q (1) * q (3)) - Accelerometer (1)
2 * (q (1) * q (2) + q (3) * q (4)) - Accelerometer (2)
2 * (0.5 - q (2) ^ 2 - q (3) ^ 2) - Accelerometer (3)
2 * b (2) * (0.5 - q (3) ^ 2 - q (4) ^ 2) + 2 * b (4) * (q (2) * q (4) - q (1) * q (3)) - Magnetometer
2 * b (2) * (q (2) * q (3) - q (1) * q (4)) + 2 * b (4) * (q (1) * q (2) + q (3) * q (4)) - Magnetometer
2 * b (2) * (q (1) * q (3) + q (2) * q (4)) + 2 * b (4) * (0.5 - q (2) ^ 2 - q (3) ^ 2) - Magnetometer
)
```

J =

$$\begin{pmatrix} -2 * q(3) & 2 * q(4) & -2 * q(1) \\ 2 * q(2) & 2 * q(1) & 2 * q(4) \\ 0 & -4 * q(2) & -4 * q(3) \\ -2 * b(4) * q(3) & 2 * b(4) * q(4) & -4 * b(2) * q(3) - 2 * b(4) * q(1) \\ -2 * b(2) * q(4) + 2 * b(4) * q(2) & 2 * b(2) * q(3) + 2 * b(4) * q(1) & 2 * b(2) * q(2) + 2 * b(4) * q(3) \\ 2 * b(2) * q(3) & 2 * b(2) * q(4) - 4 * b(4) * q(2) & 2 * b(2) * q(1) - 4 * b(4) * q(3) \end{pmatrix}$$

step = (J' * F);

■ **% normalise step magnitude**

step = step / norm (step);

■ **% Compute rate of change of quaternion**

qDot =

0.5 * quaternionProd (q, [0 Gyroscope (1) Gyroscope (2) Gyroscope (3)]) - obj.Beta * step';

■ **% Integrate to yield quaternion**

q = q + qDot * obj.SamplePeriod;

■ **% normalise quaternion**

obj.Quaternion = q / norm (q);

■ IMU

```
function obj=UpdateIMU(obj,Gyroscope,Accelerometer)
q=obj.Quaternion;% short name local variable for readability

% Normalise accelerometer measurement
if(norm(Accelerometer)==0),return;end % handle NaN
Accelerometer=Accelerometer/norm(Accelerometer);% normalise magnitude

% Gradient decent algorithm corrective step
F=[2*(q(2)*q(4)-q(1)*q(3))-Accelerometer(1) 2*(q(1)*q(2)+q(3)*q(4))-Accelerometer(2) 2*(0.5-q(2)^2-q(3)^2)-Accelerometer(3)];
J=[-2*q(3),2*q(4),-2*q(1),2*q(2) 2*q(2),2*q(1),2*q(4),2*q(3) 0,-4*q(2),-4*q(3),0];
step=(J\F);
step=step/norm(step);% normalise step magnitude

% Compute rate of change of quaternion
qDot=0.5*quaternProd(q,[0 Gyroscope(1) Gyroscope(2) Gyroscope(3)])-obj.Beta*step';

% Integrate to yield quaternion
q=q+qDot*obj.SamplePeriod;
obj.Quaternion=q/norm(q);% normalise quaternion
end
```

Kalman Filter

■ System Model

$$\mathbf{C}_1 = -\sin[\theta]$$

$$\mathbf{C}_2 = \sin[\phi] \cos[\theta]$$

$$\mathbf{C}_3 = \cos[\phi] \cos[\theta]$$

$$\dot{\mathbf{C}}_1 = -\dot{\theta} \cos[\theta] = \omega_z \mathbf{C}_2 - \omega_y \mathbf{C}_3$$

$$\dot{\mathbf{C}}_2 = \dot{\phi} \cos[\phi] \cos[\theta] - \dot{\theta} \sin[\phi] \sin[\theta] = -\omega_z \mathbf{C}_1 + \omega_x \mathbf{C}_3$$

$$\dot{\mathbf{C}}_3 = -\dot{\phi} \sin[\phi] \cos[\theta] - \dot{\theta} \cos[\phi] \sin[\theta] = \omega_y \mathbf{C}_1 - \omega_x \mathbf{C}_2$$

$$\begin{pmatrix} \dot{\mathbf{C}}_1 \\ \dot{\mathbf{C}}_2 \\ \dot{\mathbf{C}}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{pmatrix} \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} + \mathbf{W}$$

$$\mathbf{f}^b = \mathbf{v}^b + \omega_{nb}^b \times \mathbf{v}^b - \mathbf{g}^b = \begin{pmatrix} \dot{u} - r\omega + q\omega \\ \dot{v} + r\omega - p\omega \\ \dot{w} - qu + pv \end{pmatrix} + \begin{pmatrix} g \sin[\theta] \\ -g \sin[\phi] \cos[\theta] \\ -g \cos[\phi] \cos[\theta] \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{f}_x \\ \mathbf{f}_y \\ \mathbf{f}_z \end{pmatrix} = \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} - \begin{pmatrix} r\omega - q\omega \\ -r\omega + p\omega \\ qu - pv \end{pmatrix} + \begin{pmatrix} +g \sin[\theta] \\ -g \sin[\phi] \cos[\theta] \\ -g \cos[\phi] \cos[\theta] \end{pmatrix}$$

$$\approx -g \begin{pmatrix} -\sin[\theta] \\ \sin[\phi] \cos[\theta] \\ \cos[\phi] \cos[\theta] \end{pmatrix} = -g \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix}$$

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = -g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} + v$$

Kalman Filter

■ Gyro Angler Rate (p, q, r) → Euler Angle 변화율 ($\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$)

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin[\phi] \tan[\theta] & \cos[\phi] \tan[\theta] \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] / \cos[\theta] & \cos[\phi] / \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \phi + dt * \dot{\phi} \\ \theta + dt * \dot{\theta} \\ \psi + dt * \dot{\psi} \end{pmatrix}$$

■ Accelerometer (f_x , f_y , f_z)

Linear Rate (u, v, w), Linear Acceleration (ü, v̈, ẅ), Angler Rate (p, q, r), Gravity (g)

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} + g \begin{pmatrix} \sin[\theta] \\ -\cos[\theta] \sin[\phi] \\ -\cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\therefore \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = g \begin{pmatrix} \sin[\theta] \\ -\cos[\theta] \sin[\phi] \\ -\cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\theta = \text{ArcSin}\left[\frac{f_x}{g}\right]$$

$$\phi = \text{ArcSin}\left[-\frac{f_y}{g \cos[\theta]}\right]$$

■ Sensor Fusion

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{pmatrix} \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}_{k+1} = \left(\mathbf{I} + \Delta t \cdot \frac{1}{2} \begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{pmatrix} \right) \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}_k$$

$$\mathbf{A} = \mathbf{I} + \Delta t \cdot \frac{1}{2} \begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] + \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ \sin\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] - \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] + \sin\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ \cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] - \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] \end{pmatrix}$$

■ Euler Angle

ϕ around $\hat{x}_B$, θ around $\hat{y}_B$, ψ around $\hat{z}_B$

$$\phi = \text{ArcTan}[1 - 2 (q_2^2 + q_3^2), 2 (q_3 q_4 + q_1 q_2)]$$

$$\theta = -\text{ArcSin}[2 (q_2 q_4 - q_1 q_3)]$$

$$\psi = \text{ArcTan}[1 - 2 (q_3^2 + q_4^2), 2 (q_2 q_3 + q_1 q_4)]$$

■ Algorithm start

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0.0001 & 0 & 0 & 0 \\ 0 & 0.0001 & 0 & 0 \\ 0 & 0 & 0.0001 & 0 \\ 0 & 0 & 0 & 0.0001 \end{pmatrix}$$

$$R = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

■ KF

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = x$$

$$z = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] + \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ \sin\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] - \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] + \sin\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ \cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] - \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] \end{pmatrix}$$

$$xp = A \cdot x$$

$$A = I + \Delta t \cdot \frac{1}{2} \begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{pmatrix}$$

$$Pp = A \cdot P \cdot A^T + Q$$

$$K = Pp \cdot H^T \cdot \text{Inverse}[H \cdot Pp \cdot H^T + R]$$

$$x = xp + K \cdot (z - H \cdot xp)$$

$$P = Pp - K \cdot H \cdot Pp$$

$$\begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} \text{ArcTan}[1 - 2 (q_2^2 + q_3^2), 2 (q_3 q_4 + q_1 q_2)] \\ -\text{ArcSin}[2 (q_2 q_4 - q_1 q_3)] \\ \text{ArcTan}[1 - 2 (q_3^2 + q_4^2), 2 (q_2 q_3 + q_1 q_4)] \end{pmatrix}$$

Essential Kalman Filter

$$x = \begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix}$$

$$\text{rates} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\varphi} \end{pmatrix} &= \begin{pmatrix} 1 & \sin[\phi] \tan[\theta] & \cos[\phi] \tan[\theta] \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] / \cos[\theta] & \cos[\phi] / \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} + w \\ &= \begin{pmatrix} p + r \cos[\phi] \tan[\theta] + q \sin[\phi] \tan[\theta] \\ q \cos[\phi] - r \sin[\phi] \\ r \cos[\phi] \sec[\theta] + q \sec[\theta] \sin[\phi] \end{pmatrix} + w \\ &= f[x] + w \end{aligned}$$

$$\begin{aligned} z &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix} + v \\ &= H x + v \end{aligned}$$

■ Algorithm start

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial \phi} & \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \varphi} \\ \frac{\partial f_2}{\partial \phi} & \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \varphi} \\ \frac{\partial f_3}{\partial \phi} & \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial \varphi} \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0.0001 & 0 & 0 \\ 0 & 0.0001 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

■ EKF

$$\begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix} = \mathbf{x}$$

$$\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{r} \end{pmatrix} = \text{rates}$$

$$\mathbf{A} = \text{A}[\mathbf{x}, \text{rates}, \text{dt}]$$

$$\mathbf{A} =$$

$$\begin{pmatrix} q \cos[\phi] \tan[\theta] - r \sin[\phi] \tan[\theta] & q \sin[\phi] \sec[\theta]^2 + r \cos[\phi] \sec[\theta]^2 & 0 \\ -q \sin[\phi] - r \cos[\phi] & 0 & 0 \\ q \cos[\phi] \sec[\theta] - r \sin[\phi] \sec[\theta] & q \sin[\phi] \sec[\theta] \tan[\theta] + r \cos[\phi] \sec[\theta] \tan[\theta] & 0 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{I} + \Delta t * \mathbf{A}$$

$$\mathbf{x}_p = \mathbf{f}[\mathbf{x}, \text{rates}, \text{dt}]$$

$$\mathbf{x}_p = \mathbf{f}[\mathbf{x}, \text{rates}, \text{dt}] = \begin{pmatrix} \mathbf{p} + r \cos[\phi] \tan[\theta] + q \sin[\phi] \tan[\theta] \\ q \cos[\phi] - r \sin[\phi] \\ r \cos[\phi] \sec[\theta] + q \sec[\theta] \sin[\phi] \end{pmatrix}$$

$$\mathbf{P}_p = \mathbf{A} \cdot \mathbf{P} \cdot \mathbf{A}^T + \mathbf{Q}$$

$$\mathbf{K} = \mathbf{P}_p \cdot \mathbf{H}^T \cdot \text{Inverse}[\mathbf{H} \cdot \mathbf{P}_p \cdot \mathbf{H}^T + \mathbf{R}]$$

$$\mathbf{x} = \mathbf{x}_p + \mathbf{K} \cdot (\mathbf{z} - \mathbf{H} * \mathbf{x}_p)$$

$$\mathbf{P} = \mathbf{P}_p - \mathbf{K} \cdot \mathbf{H} \cdot \mathbf{P}_p$$

Rotation Matrix

동차변환을 이용한 좌표변환