

Euler Angles and Rotation Matrix

■ Rotation Matrix

Direction Cosine Matrix

$$\begin{pmatrix} n & o & a \end{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix}$$

■ unit column vector

$$\mathbf{n} = \begin{pmatrix} n_x & n_y & n_z \end{pmatrix}^T$$

$$\mathbf{o} = \begin{pmatrix} o_x & o_y & o_z \end{pmatrix}^T$$

$$\mathbf{a} = \begin{pmatrix} a_x & a_y & a_z \end{pmatrix}^T$$

$$\mathbf{R} = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}$$

$$\mathbf{n} \cdot \mathbf{o} = 0$$

$$\mathbf{o} \cdot \mathbf{a} = 0$$

$$\mathbf{a} \cdot \mathbf{n} = 0$$

$$|\mathbf{n}| = 1$$

$$|\mathbf{o}| = 1$$

$$|\mathbf{a}| = 1$$

■ Orthogonal Matrix

$$\mathbf{R}^T \mathbf{R} = \mathbf{I}$$

$$\mathbf{R}^T = \mathbf{R}^{-1}$$

■ ZYX (Roll-Pitch-Yaw) Angles

X - axis (Yaw) $\phi \rightarrow R_x[\phi]$

Y - axis (Pitch) $\theta \rightarrow R_y[\theta]$

Z - axis (Roll) $\psi \rightarrow R_z[\psi]$

$$R_{zyx} = R_z[\psi] R_y[\theta] R_x[\phi]$$

$$\begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

■ XYZ Angles

$$R_{xyz} = R_x[\phi] R_y[\theta] R_z[\psi]$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] \cos[\psi] & -\cos[\theta] \sin[\psi] & \sin[\theta] \\ \cos[\psi] \sin[\theta] \sin[\phi] + \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\theta] \sin[\phi] \\ -\cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] & \cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

■ Rotation Matrix and Small Angles

$$\cos[\alpha] \approx 1, \sin[\alpha] \approx \alpha$$

$$\mathbf{R}_{zyx} = \mathbf{R}_z[\psi] \mathbf{R}_y[\theta] \mathbf{R}_x[\phi]$$

$$\begin{pmatrix} 1 & -\psi & 0 \\ \psi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ -\theta & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\phi \\ 0 & \phi & 1 \end{pmatrix} // \text{MatrixForm}$$

$$\mathbf{R}_{zyx} = \begin{pmatrix} 1 & \theta\phi - \psi & \theta + \phi\psi \\ \psi & 1 + \theta\phi\psi & -\phi + \theta\psi \\ -\theta & \phi & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\phi & \theta \\ \phi & 1 & -\psi \\ -\theta & \psi & 1 \end{pmatrix}$$

$$\mathbf{R}_{xyz} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\phi \\ 0 & \phi & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ -\theta & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\psi & 0 \\ \psi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} // \text{MatrixForm}$$

$$\mathbf{R}_{xyz} = \begin{pmatrix} 1 & -\psi & \theta \\ \theta\phi + \psi & 1 - \theta\phi\psi & -\phi \\ -\theta + \phi\psi & \phi + \theta\psi & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\psi & \theta \\ \psi & 1 & -\phi \\ -\theta & \phi & 1 \end{pmatrix}$$

Rotation Matrix to Euler Angles

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

■ ZYX (Roll-Pitch-Yaw) Angles

■ $\theta = \{-\pi/2, \pi/2\}$

$$\psi = \text{ArcTan}[r_{11}, r_{21}]$$

$$\theta = \text{ArcSin}[-r_{31}]$$

$$\phi = \text{ArcTan}[r_{33}, r_{32}]$$

■ $\theta = \{\pi/2, 3\pi/2\}$

$$\psi = \text{ArcTan}[-r_{11}, -r_{21}]$$

$$\theta = \text{ArcSin}[-r_{31}]$$

$$\phi = \text{ArcTan}[-r_{33}, -r_{32}]$$

■ θ (Pitch) = $\pi/2$, $\text{Cos}[\theta] \approx 0$

$$r_{23} - r_{12} = (\text{Sin}[\theta] + 1) \text{Sin}[\psi - \phi]$$

$$r_{13} + r_{22} = (\text{Sin}[\theta] + 1) \text{Cos}[\psi - \phi]$$

$$r_{23} + r_{12} = (\text{Sin}[\theta] - 1) \text{Sin}[\psi + \phi]$$

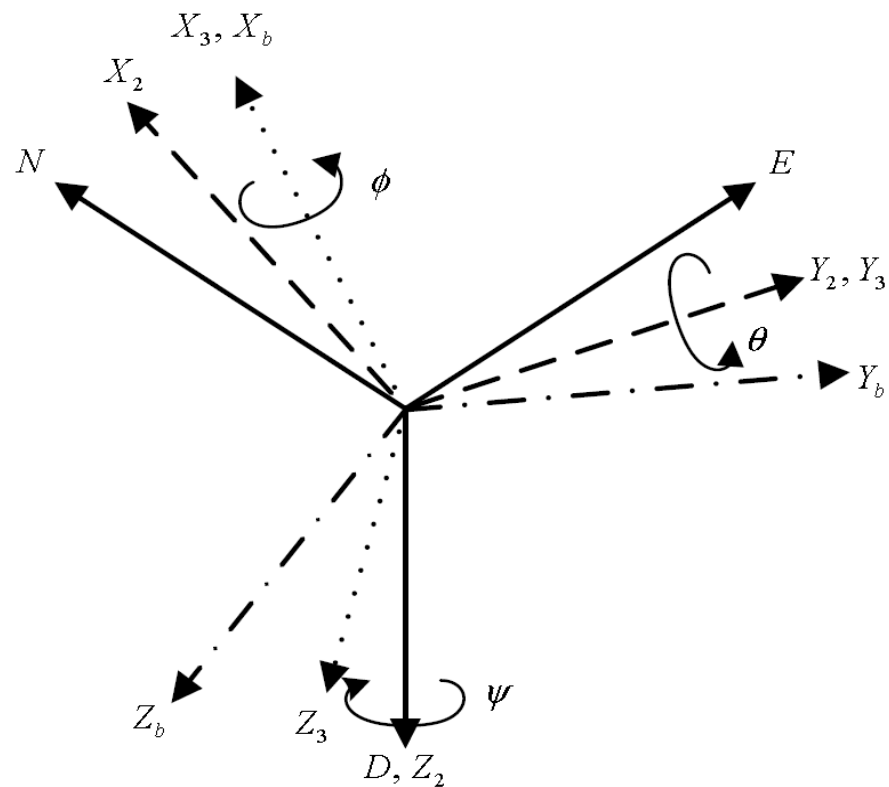
$$r_{13} - r_{22} = (\text{Sin}[\theta] - 1) \text{Cos}[\psi + \phi]$$

$$\psi = \text{ArcTan}[r_{13} - r_{22}, r_{23} + r_{12}]$$

$$\theta = \text{ArcSin}[-r_{31}]$$

$$\phi = \psi - \text{ArcTan}[r_{13} + r_{22}, r_{23} - r_{12}]$$

Euler Angle



$$\mathbf{C}_n^b = \mathbf{C}_3^b[\phi] \mathbf{C}_2^3[\theta] \mathbf{C}_n^2[\psi]$$

$$\mathbf{R}_{zyx} = \mathbf{R}_z[\psi] \mathbf{R}_y[\theta] \mathbf{R}_x[\phi]$$

$$\begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \text{Cos}[\theta] \text{Cos}[\psi] & \text{Cos}[\psi] \text{Sin}[\theta] \text{Sin}[\phi] - \text{Cos}[\phi] \text{Sin}[\psi] & \text{Cos}[\phi] \text{Cos}[\psi] \text{Sin}[\theta] + \text{Sin}[\phi] \text{Sin}[\psi] \\ \text{Cos}[\theta] \text{Sin}[\psi] & \text{Cos}[\phi] \text{Cos}[\psi] + \text{Sin}[\theta] \text{Sin}[\phi] \text{Sin}[\psi] & -\text{Cos}[\psi] \text{Sin}[\phi] + \text{Cos}[\phi] \text{Sin}[\theta] \text{Sin}[\psi] \\ -\text{Sin}[\theta] & \text{Cos}[\theta] \text{Sin}[\phi] & \text{Cos}[\theta] \text{Cos}[\phi] \end{pmatrix}$$

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \text{C}_3^b[\phi] \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \text{C}_3^b[\phi] \text{C}_2^3[\theta] \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \text{Sin}[\phi] \text{Tan}[\theta] & \text{Cos}[\phi] \text{Tan}[\theta] \\ 0 & \text{Cos}[\phi] & -\text{Sin}[\phi] \\ 0 & \text{Sin}[\phi] \text{Sec}[\theta] & \text{Cos}[\phi] \text{Sec}[\theta] \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

■ Derivation

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[\phi] & -\text{Sin}[\phi] \\ 0 & \text{Sin}[\phi] & \text{Cos}[\phi] \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[\phi] & -\text{Sin}[\phi] \\ 0 & \text{Sin}[\phi] & \text{Cos}[\phi] \end{pmatrix} \cdot \begin{pmatrix} \text{Cos}[\theta] & 0 & \text{Sin}[\theta] \\ 0 & 1 & 0 \\ -\text{Sin}[\theta] & 0 & \text{Cos}[\theta] \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot{\phi} + \dot{\psi} \text{Sin}[\theta] \\ \text{Cos}[\phi] \dot{\theta} - \text{Cos}[\theta] \dot{\psi} \text{Sin}[\phi] \\ \text{Cos}[\theta] \text{Cos}[\phi] \dot{\psi} + \dot{\theta} \text{Sin}[\phi] \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \text{Sin}[\phi] \text{Tan}[\theta] & \text{Cos}[\phi] \text{Tan}[\theta] \\ 0 & \text{Cos}[\phi] & -\text{Sin}[\phi] \\ 0 & \text{Sin}[\phi] \text{Sec}[\theta] & \text{Cos}[\phi] \text{Sec}[\theta] \end{pmatrix} \cdot \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \omega_x + \text{Sin}[\phi] \omega_y \text{Tan}[\theta] + \text{Cos}[\phi] \omega_z \text{Tan}[\theta] \\ \text{Cos}[\phi] \omega_y - \text{Sin}[\phi] \omega_z \\ \text{Sec}[\theta] \text{Sin}[\phi] \omega_y + \text{Cos}[\phi] \text{Sec}[\theta] \omega_z \end{pmatrix}$$

$$\text{Solve}[\{\dot{\phi} == \omega_x + \text{Sin}[\phi] \omega_y \text{Tan}[\theta] + \text{Cos}[\phi] \omega_z \text{Tan}[\theta], \dot{\theta} == \text{Cos}[\phi] \omega_y - \text{Sin}[\phi] \omega_z, \dot{\psi} == \text{Sec}[\theta] \text{Sin}[\phi] \omega_y + \text{Cos}[\phi] \text{Sec}[\theta] \omega_z\}, \{\omega_x, \omega_y, \omega_z\}]$$

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \dot{\phi} - \dot{\psi} \text{Sin}[\theta] \\ \frac{1}{2} (2 \text{Cos}[\phi] \dot{\theta} - \dot{\psi} \text{Sin}[\theta - \phi] + \dot{\psi} \text{Sin}[\theta + \phi]) \\ \frac{1}{2} (\text{Cos}[\theta - \phi] \dot{\psi} + \text{Cos}[\theta + \phi] \dot{\psi} - 2 \dot{\theta} \text{Sin}[\phi]) \end{pmatrix} = \begin{pmatrix} \dot{\phi} - \dot{\psi} \text{Sin}[\theta] \\ \text{Cos}[\phi] \dot{\theta} + \text{Cos}[\theta] \dot{\psi} \text{Sin}[\phi] \\ \text{Cos}[\theta] \text{Cos}[\phi] \dot{\psi} - \dot{\theta} \text{Sin}[\phi] \end{pmatrix}$$

Kalman Filter

■ System Model

$$C_1 = -\sin[\theta]$$

$$C_2 = \sin[\phi] \cos[\theta]$$

$$C_3 = \cos[\phi] \cos[\theta]$$

$$\dot{C}_1 = -\dot{\theta} \cos[\theta] = \omega_z C_2 - \omega_y C_3$$

$$\dot{C}_2 = \dot{\phi} \cos[\phi] \cos[\theta] - \dot{\theta} \sin[\phi] \sin[\theta] = -\omega_z C_1 \omega_x C_3$$

$$\dot{C}_3 = -\dot{\phi} \sin[\phi] \cos[\theta] - \dot{\theta} \cos[\phi] \sin[\theta] = \omega_y C_1 - \omega_x C_2$$

$$\begin{pmatrix} \dot{C}_1 \\ \dot{C}_2 \\ \dot{C}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} + W$$

$$f^b = v^{-b} + \omega_{nb}^b \times v^b - g^{-b} = \begin{pmatrix} \dot{u} - ru + qw \\ \dot{v} + ru - pw \\ \dot{w} - qu + pv \end{pmatrix} + \begin{pmatrix} g \sin[\theta] \\ -g \sin[\phi] \cos[\theta] \\ -g \cos[\phi] \cos[\theta] \end{pmatrix}$$

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} - \begin{pmatrix} ru - q\omega \\ -ru + p\omega \\ qu - pv \end{pmatrix} + \begin{pmatrix} +g \sin[\theta] \\ -g \sin[\phi] \cos[\theta] \\ -g \cos[\phi] \cos[\theta] \end{pmatrix}$$

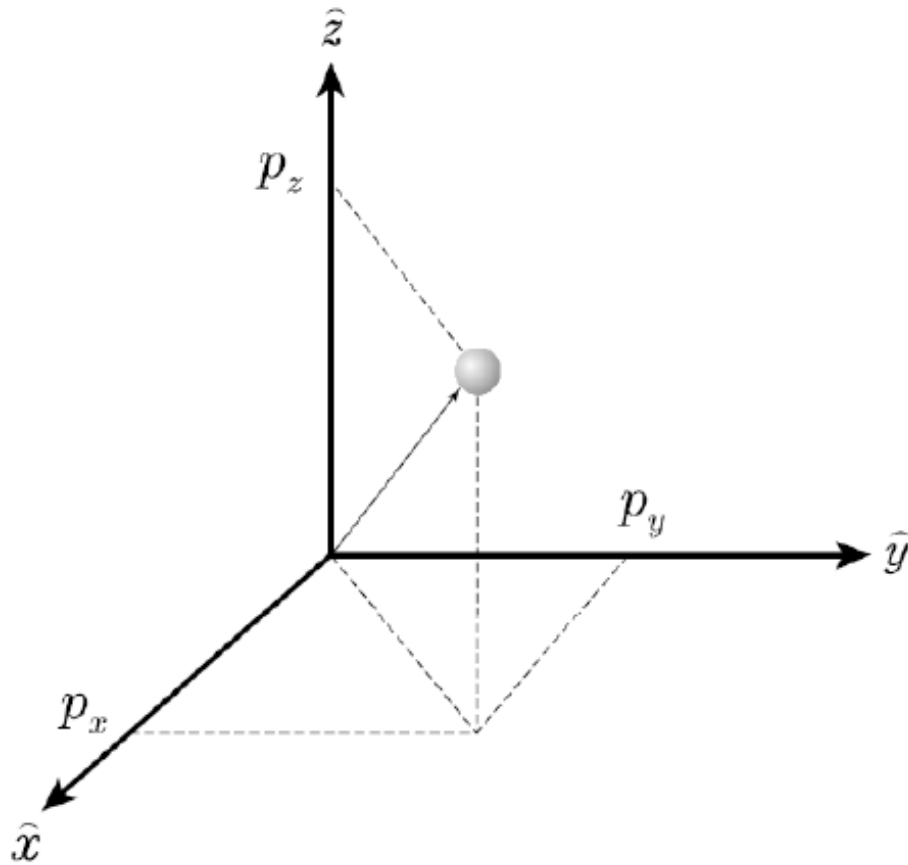
$$\approx -g \begin{pmatrix} -\sin[\theta] \\ \sin[\phi] \cos[\theta] \\ \cos[\phi] \cos[\theta] \end{pmatrix} = -g \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{f}_x \\ \mathbf{f}_y \\ \mathbf{f}_z \end{pmatrix} = -\mathbf{g} \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{pmatrix} + \mathbf{V}$$

Analysis on Position And Orientation

■ Cartesian coordinate (직교좌표계)

unit vector

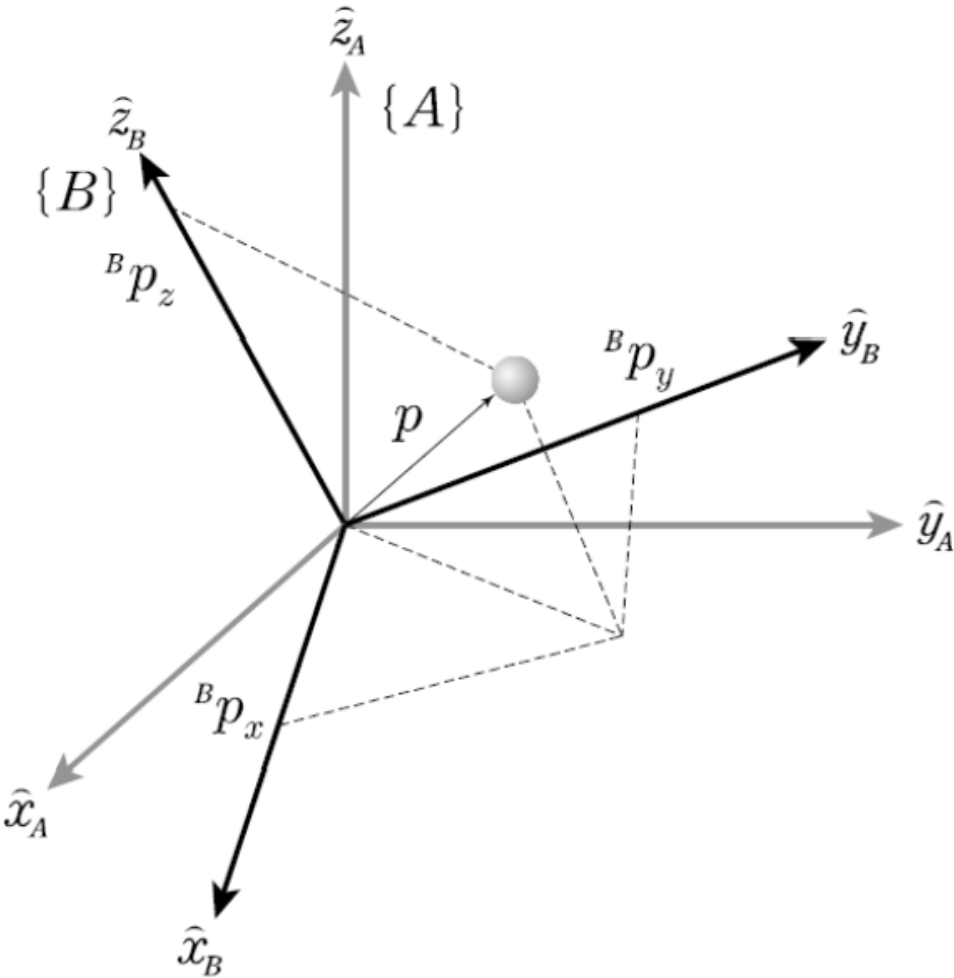


■ One Point P Vector

$$\mathbf{P} = P \cdot \hat{x} + P \cdot \hat{y} + P \cdot \hat{z} = p_x \hat{x} + p_y \hat{y} + p_z \hat{z}$$

$$\text{Direction Vector : } \mathbf{P} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

위치 표현



좌표계 {A}, {B} 에서 본 점 P의 위치

x_B, y_B, z_B 를 좌표계 {A} 의 좌표성분으로 표시하면

$$x_B = x_B \cdot x_A + x_B \cdot y_A + x_B \cdot z_A$$

$$x_B = \begin{pmatrix} x_B \cdot x_A \\ x_B \cdot y_A \\ x_B \cdot z_A \end{pmatrix}$$

좌표계 {A} 에서 본 벡터의 방향, **Direction Vector** (방향 벡터)

Dirction Cosine (: 단위벡터의 내적→두 벡터 사이 가의 코사인 값)

$$\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z$$

$$\alpha = \text{Cos}[a] = \frac{\mathbf{v} \cdot \mathbf{e}_x}{|\mathbf{v}|} = \frac{v_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}}$$

$$\beta = \text{Cos}[b] = \frac{\mathbf{v} \cdot \mathbf{e}_y}{|\mathbf{v}|} = \frac{v_y}{\sqrt{v_x^2 + v_y^2 + v_z^2}}$$

$$\gamma = \text{Cos}[c] = \frac{\mathbf{v} \cdot \mathbf{e}_z}{|\mathbf{v}|} = \frac{v_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}}$$

$$\text{Cos}[a]^2 + \text{Cos}[b]^2 + \text{Cos}[c]^2 = 1$$

상대 개념 위치

상대위치표현 ${}^A\mathbf{P}$, ${}^B\mathbf{P}$ 의 표현

$$\begin{aligned} \mathbf{P} &= {}^A\mathbf{P} = {}^A p_x \hat{x}_A + {}^A p_y \hat{y}_A + {}^A p_z \hat{z}_A \\ &= {}^B\mathbf{P} = {}^B p_x \hat{x}_B + {}^B p_y \hat{y}_B + {}^B p_z \hat{z}_B \end{aligned}$$

$$\begin{cases} {}^A p_x, {}^A p_y, {}^A p_z : \mathbf{P} \text{의 } \hat{x}_A, \hat{y}_A, \hat{z}_A \text{ 방향 성분} \\ {}^B p_x, {}^B p_y, {}^B p_z : \mathbf{P} \text{의 } \hat{x}_B, \hat{y}_B, \hat{z}_B \text{ 방향 성분} \end{cases}$$

Rotation Matrix ${}^A_B R$

$$\hat{x}_B = \hat{x}_B \cdot \hat{x}_A + \hat{x}_B \cdot \hat{y}_A + \hat{x}_B \cdot \hat{z}_A$$

$$\hat{y}_B = \hat{y}_B \cdot \hat{x}_A + \hat{y}_B \cdot \hat{y}_A + \hat{y}_B \cdot \hat{z}_A$$

$$\hat{z}_B = \hat{z}_B \cdot \hat{x}_A + \hat{z}_B \cdot \hat{y}_A + \hat{z}_B \cdot \hat{z}_A$$

$${}^A_B R = \begin{pmatrix} \hat{x}_B & \hat{y}_B & \hat{z}_B \end{pmatrix} = \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix}$$

$\hat{x}_B \ \hat{y}_B \ \hat{z}_B$ 를 각각 $\{A\}$ 좌표성분 $(\hat{x}_A \ \hat{y}_A \ \hat{z}_A)$ 으로 표현한 식들의 조합으로 $\{A\}$ 에서 본 $\{B\}$ 의 표현식

좌표계 $\{A\}$ 와 $\{B\}$ 사이의 변환 행렬으로 $\{B\}$ 의 $\{A\}$ 에 대한 상대관계 또는 상대 회전량

물체의 방위 (상대적 회전량) 를 나타냄

$$\blacksquare {}^A\mathbf{P} = {}^A_B R \cdot {}^B\mathbf{P}$$

$\{B\}$ 좌표계 성분을 $\{A\}$ 좌표계 성분으로 변환

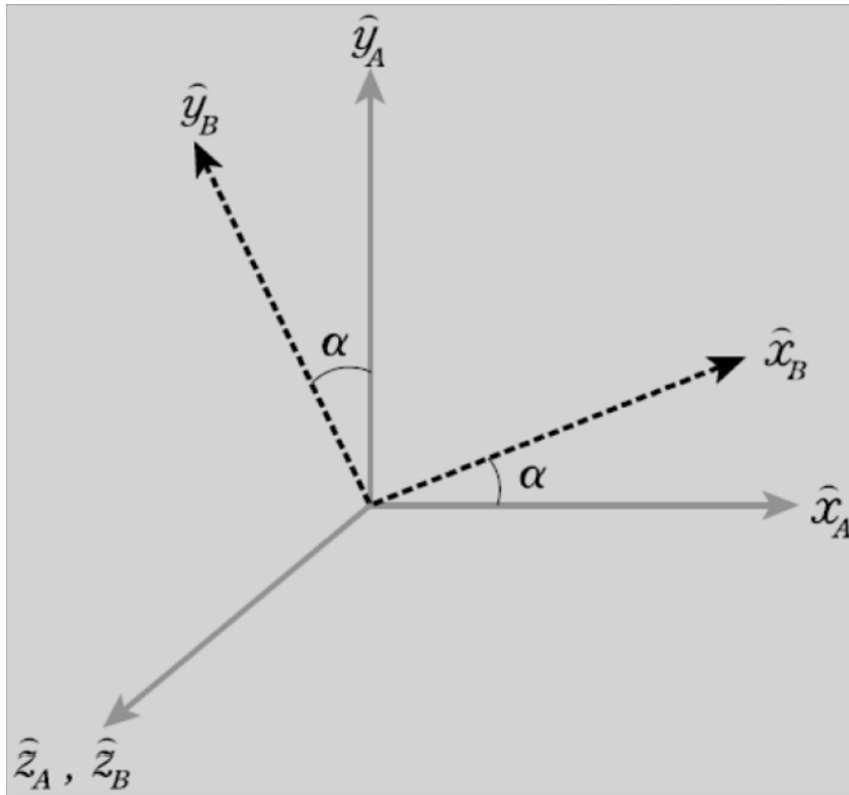
$$\begin{aligned}
{}^A\mathbf{P} &= \begin{pmatrix} {}^A p_x \\ {}^A p_y \\ {}^A p_z \end{pmatrix} = \begin{pmatrix} {}^A\mathbf{P} \cdot \hat{x}_A \\ {}^A\mathbf{P} \cdot \hat{y}_A \\ {}^A\mathbf{P} \cdot \hat{z}_A \end{pmatrix} = \begin{pmatrix} \mathbf{P} \cdot \hat{x}_A \\ \mathbf{P} \cdot \hat{y}_A \\ \mathbf{P} \cdot \hat{z}_A \end{pmatrix} = \begin{pmatrix} {}^B\mathbf{P} \cdot \hat{x}_A \\ {}^B\mathbf{P} \cdot \hat{y}_A \\ {}^B\mathbf{P} \cdot \hat{z}_A \end{pmatrix} = \begin{pmatrix} ({}^B p_x \hat{x}_B + {}^B p_y \hat{y}_B + {}^B p_z \hat{z}_B) \cdot \hat{x}_A \\ ({}^B p_x \hat{x}_B + {}^B p_y \hat{y}_B + {}^B p_z \hat{z}_B) \cdot \hat{y}_A \\ ({}^B p_x \hat{x}_B + {}^B p_y \hat{y}_B + {}^B p_z \hat{z}_B) \cdot \hat{z}_A \end{pmatrix} \\
&= \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix} \cdot \begin{pmatrix} {}^B p_x \\ {}^B p_y \\ {}^B p_z \end{pmatrix}
\end{aligned}$$

$${}^A\mathbf{P} = {}^A_B\mathbf{R} \cdot {}^B\mathbf{P}$$

→ 좌표계 {B}에 대해 표현된 점P의 위치 ${}^B\mathbf{P}$ 가 좌표계 {A}의 좌표값 ${}^A\mathbf{P}$ 로 회전행렬 ${}^A_B\mathbf{R}$ 을 통하여 변환됨을 의미

$\Rightarrow {}^A_B\mathbf{R}$: Transformation Matrix (변환행렬) 라고 정의
좌표계 {A}에 대한 좌표계 {B}의 회전행렬

방위의 표현: 회전형렬



좌표계 {A} 가 $\hat{z}_A$ 축을 기준으로 α 만큼 회전한 좌표계를 {B} 라고 할 경우 회전형렬 $\hat{\mathbf{R}}$ 을 구하라.

$$\begin{aligned}
{}^A_B\mathbf{R} &= \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix} = \begin{pmatrix} \cos[\alpha] & \cos[90^\circ + \alpha] & \cos[90^\circ] \\ \cos[90^\circ - \alpha] & \cos[\alpha] & \cos[90^\circ] \\ \cos[90^\circ] & \cos[90^\circ] & \cos[0^\circ] \end{pmatrix} \\
&= \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \text{Rot}(\hat{z}, \alpha)
\end{aligned}$$

$$\text{Rot}(\hat{z}, \psi) = \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{Rot}(\hat{y}, \theta) = \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix}$$

$$\text{Rot}(\hat{x}, \phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix}$$

■ 회전행렬의 성질 (${}^B_A\mathbf{R} = {}^A\mathbf{R}^{-1} = {}^A\mathbf{R}^T$)

좌표계 {B} 에 대한 좌표계 {A} 의 회전행렬 : ${}^B_A\mathbf{R}$

좌표계 {A} 에 대한 좌표계 {B} 의 회전행렬 : ${}^A_B\mathbf{R}$

$${}^B_A\mathbf{R} = \begin{pmatrix} \hat{x}_A \cdot \hat{x}_B & \hat{y}_A \cdot \hat{x}_B & \hat{z}_A \cdot \hat{x}_B \\ \hat{x}_A \cdot \hat{y}_B & \hat{y}_A \cdot \hat{y}_B & \hat{z}_A \cdot \hat{y}_B \\ \hat{x}_A \cdot \hat{z}_B & \hat{y}_A \cdot \hat{z}_B & \hat{z}_A \cdot \hat{z}_B \end{pmatrix} \iff {}^A_B\mathbf{R} = \begin{pmatrix} \hat{x}_B \cdot \hat{x}_A & \hat{y}_B \cdot \hat{x}_A & \hat{z}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A & \hat{y}_B \cdot \hat{y}_A & \hat{z}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A & \hat{y}_B \cdot \hat{z}_A & \hat{z}_B \cdot \hat{z}_A \end{pmatrix}$$

$${}^B_A\mathbf{R} = {}^A_B\mathbf{R}^T$$

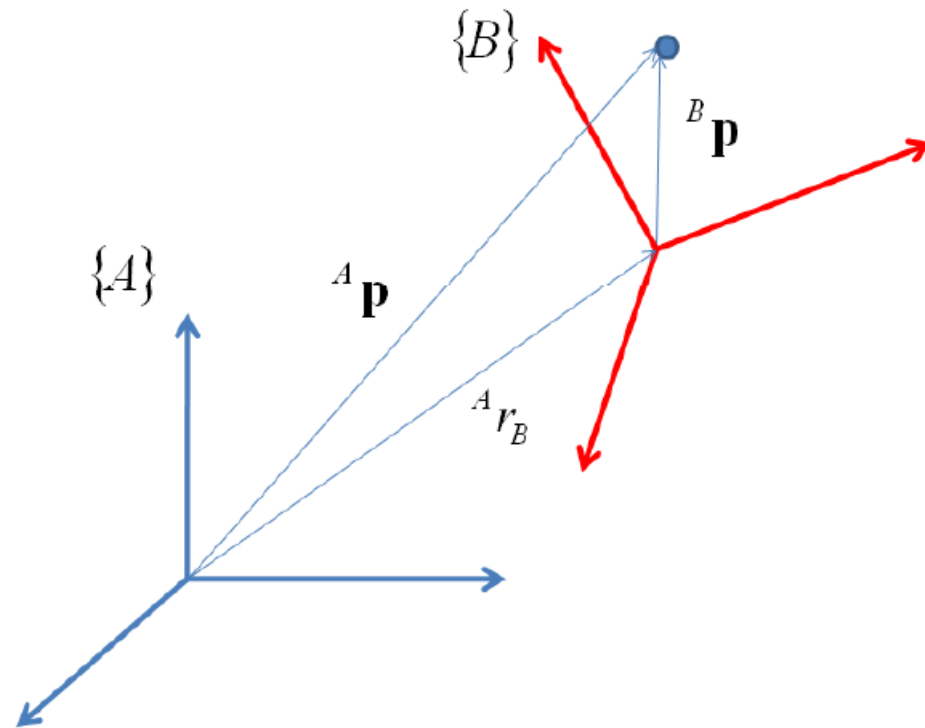
$$\begin{cases} {}^A\mathbf{P} = {}^A_B\mathbf{R} \cdot {}^B\mathbf{P} \\ {}^B\mathbf{P} = {}^B_A\mathbf{R} \cdot {}^A\mathbf{P} \end{cases} \implies {}^A\mathbf{P} = {}^A_B\mathbf{R} \cdot {}^B\mathbf{P} = {}^A_B\mathbf{R} \cdot {}^B_A\mathbf{R} \cdot {}^A\mathbf{P} \implies {}^A_B\mathbf{R} \cdot {}^B_A\mathbf{R} = \mathbf{I}$$

$${}^B_A\mathbf{R} = {}^A_B\mathbf{R}^{-1} = {}^A_B\mathbf{R}^T$$

동차변환을 이용한 좌표변환

■ ${}^A\mathbf{P} = {}^A\mathbf{r}_B + {}^A\mathbf{R}_B \cdot {}^B\mathbf{P}$

방위가 서로 다르고 원점이 ${}^A\mathbf{r}_B$ 만큼 떨어진 좌표계 $\{A\}$ 와 $\{B\}$



${}^A\mathbf{P} = {}^A\mathbf{r}_B + {}^B\mathbf{P}$ (incorrect!)

${}^A\mathbf{P}$, ${}^B\mathbf{P}$ 의 x, y, z 성분들은 서로 방위가 다른 좌표계 기준으로 표현되어 있어 직접 더해질 수 없다.

$${}^A\mathbf{P} = {}^A\mathbf{r}_B + {}^A\mathbf{R}_B \cdot {}^B\mathbf{P} \text{ (correct !)}$$

${}^A\mathbf{r}_B$: 두 좌표계 사이의 상대거리 또는 전위 또는 병진 (Translation)

${}^A\mathbf{R}_B$: 두 좌표계 사이의 상대회전 (Rotation)

$$\begin{pmatrix} {}^A\mathbf{P} \\ 1 \end{pmatrix} = \begin{pmatrix} {}^A\mathbf{R}_B & {}^A\mathbf{r}_B \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} {}^B\mathbf{P} \\ 1 \end{pmatrix}$$

$${}^A\mathbf{P} = {}^A\mathbf{T}_B \cdot {}^B\mathbf{P}$$

${}^A\mathbf{T}_B$: 동차 변환행렬 (homogeneous transformation matrix)

■ 동차 변환행렬의 구조와 표현

$${}^A\mathbf{T}_B = \begin{pmatrix} {}^A\mathbf{R}_B & {}^A\mathbf{r}_B \\ 0 & 1 \end{pmatrix}$$

좌표계 {B} 로 표현된 벡터 (${}^B\mathbf{P}$) 를 좌표계 {A} 에 대해 표현하는 좌표변환 (Coordinate Transformation) 으로서,
좌표계 {B} 의 좌표계 {A} 에 대한 회전형렬 (3×3) 과 좌표계 {A} 의 원점에서 좌표계 {B} 의 원점까지의 병진벡터 (3×1) 를 포함하는 4×4 행렬

$$\mathbf{T} = \text{Trans}[p_x, p_y, p_z] = \begin{pmatrix} 1 & 0 & 0 & {}^A p_x \\ 0 & 1 & 0 & {}^A p_y \\ 0 & 0 & 1 & {}^A p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^A\mathbf{T}_B = \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 & {}^A p_x \\ \sin[\alpha] & \cos[\alpha] & 0 & {}^A p_y \\ 0 & 0 & 1 & {}^A p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & {}^A p_x \\ 0 & 1 & 0 & {}^A p_y \\ 0 & 0 & 1 & {}^A p_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \text{Trans}[p_x, p_y, p_z] \cdot \text{Rot}[z_A, \alpha]$$

$${}^A_B\mathbf{T} = \text{Trans}[p_x, p_y, p_z] \cdot \text{Rot}[z_A, \alpha]$$

$$\text{주의} \rightarrow {}^A_B\mathbf{T} \neq \text{Rot}[z_A, \alpha] \cdot \text{Trans}[p_x, p_y, p_z]$$

역변환 (Inverse Transformation)

inverse transformation for a given ${}^A_B\mathbf{T} \Rightarrow {}^B_A\mathbf{T}$

$${}^A\mathbf{P} = {}^A_B\mathbf{T} \cdot {}^B\mathbf{P} \Leftrightarrow {}^B\mathbf{P} = {}^B_A\mathbf{T} \cdot {}^A\mathbf{P} = ({}^A_B\mathbf{T})^{-1} \cdot {}^A\mathbf{P}$$

$${}^B_A\mathbf{T} = ({}^A_B\mathbf{T})^{-1}$$

■ 유도

$${}^A\mathbf{P} = {}^A\mathbf{r}_B + {}^A_B\mathbf{R} \cdot {}^B\mathbf{P}$$

$${}^A_B\mathbf{R}^T \cdot {}^A\mathbf{P} = {}^A_B\mathbf{R}^T \cdot {}^A\mathbf{r}_B + {}^A_B\mathbf{R}^T \cdot {}^A_B\mathbf{R} \cdot {}^B\mathbf{P}$$

$${}^A_B\mathbf{R}^T \cdot {}^A\mathbf{P} = {}^A_B\mathbf{R}^T \cdot {}^A\mathbf{r}_B + {}^B\mathbf{P}$$

$${}^B\mathbf{P} = {}^A_B\mathbf{R}^T \cdot {}^A\mathbf{r}_B - {}^A_B\mathbf{R}^T \cdot {}^A\mathbf{P}$$

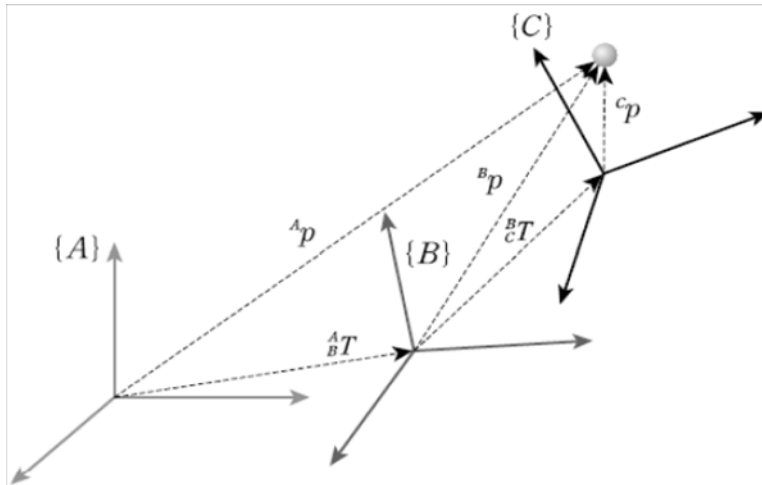
$$\begin{pmatrix} {}^B\mathbf{P} \\ 1 \end{pmatrix} = \begin{pmatrix} {}^A_B\mathbf{R}^T & -{}^A_B\mathbf{R}^T \cdot {}^A\mathbf{r}_B \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} {}^A\mathbf{P} \\ 1 \end{pmatrix}$$

$${}^B\mathbf{P} = {}^B_A\mathbf{T} \cdot {}^A\mathbf{P}$$

$${}^B_A\mathbf{T} = ({}^A_B\mathbf{T})^{-1} = \begin{pmatrix} {}^A_B\mathbf{R}^T & -{}^A_B\mathbf{R}^T \cdot {}^A\mathbf{r}_B \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

변환의 합성 (${}^A_C T = {}^A_B T \cdot {}^B_C T$)

${}^B_C T$, ${}^A_B T$ 가 주어졌을 때 점 P의 좌표계 {A}에 대한 표현 ${}^A P$



$$\begin{cases} {}^B P = {}^B_C T \cdot {}^C P \\ {}^A P = {}^A_B T \cdot {}^B P \end{cases}$$

$${}^A P = {}^A_B T \cdot {}^B P = {}^A_B T \cdot {}^B_C T \cdot {}^C P = {}^A_C T \cdot {}^C P$$

$${}^A_C T = {}^A_B T \cdot {}^B_C T$$

- 변환 방정식: 페루프 이용으로 미지의 변환을 다른 변환들 사이의 관계를 통해서 얻을 수 있다.

연속좌표변환: 절대변환과 상대변환

- 이동좌표계 (moving frame)

변환 (회전 또는 병진) 에 의해 새롭게 얻어지는 좌표계

- 고정좌표계 (fixed frame)

변환에 관계없이 처음과 동일하게 고정된 좌표계

- 절대변환 (absolute transform)

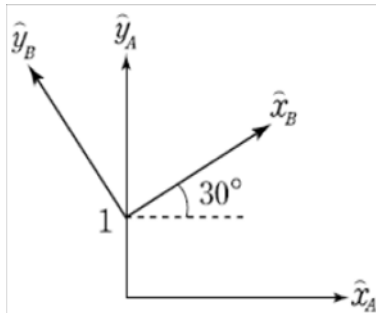
전 단계의 변환에 관계없이 후속변환을 계속 초기의 고정된 기준좌표인 고정좌표계에 대해 수행하는 것

- 상대변환 (relative transform)

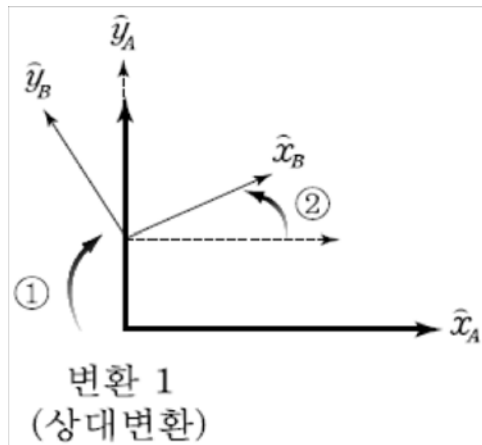
각 단계의 변환 후 얻어진 새로운 좌표계인 이동좌표계에 대하여 다음 변환이 수행 되는 것

- 좌표계{A}와{B} 사이의 변환을 나타내는 동차변환행렬

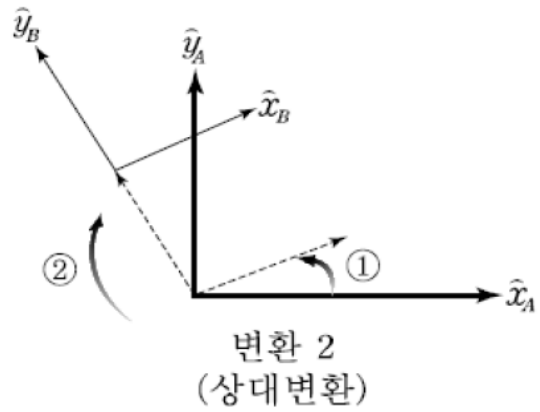
$${}^A_B T = \text{Trans}[p_x, p_y, p_z] \cdot \text{Rot}[z_A, \alpha]$$



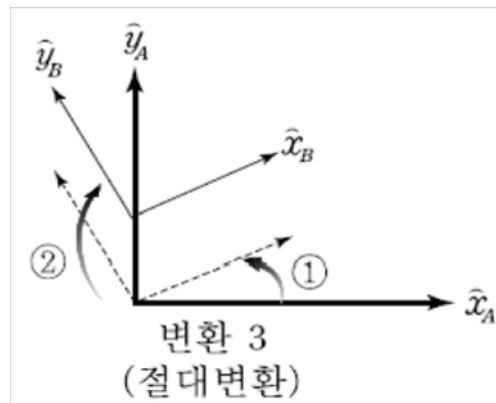
- O: 이동좌표계를 따라 병진 후 이동좌표계에 대해 회전



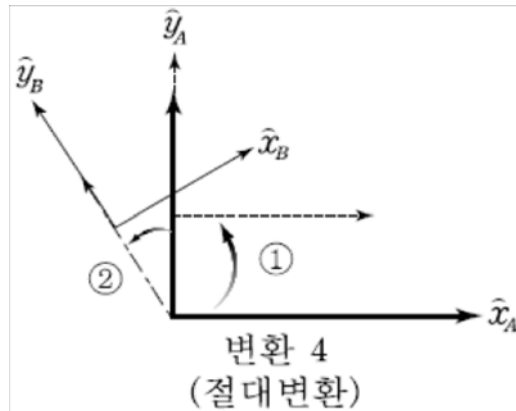
- X: 이동좌표계를 따라 회전 후 이동좌표계에 대해 병진



- O: 고정좌표계를 따라 회전 후 고정좌표계에 대해 병진



■ X: 고정좌표계를 따라 병진 후 고정좌표계에 대해 회전



■ 올바른 변환을 나타내는 조합

상대변환 : 이동좌표계에 대한 병진 → 이동좌표계에 대해 회전

절대변환 : 고정좌표계에 대한 회전 → 고정좌표계에 대해 병진

두가지 모두 동일한 결과 ${}^A_B T$ 를 나타냄

$${}^A_B T = \text{Trans}[p_x, p_y, p_z] \cdot \text{Rot}[\alpha, \beta, \gamma]$$

∴ 상대변환의 구현 (post - multiply)

회전과 병진을 나타내는 변환행렬이 앞에서부터 뒤로 순차적으로 곱해져야 함

∴ 절대변환의 구현 (pre - multiply)

반대로 해당 변환행렬이 뒤에서부터 앞으로 순차적으로 곱해져야 함

$${}^A r_D = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$${}^B\mathbf{r}_D = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = {}^B\mathbf{r}_D'$$

$${}^A\mathbf{r}_D' = {}^A\mathbf{T}_B \cdot {}^B\mathbf{r}_D'$$

$${}^A\mathbf{r}_C' = {}^A\mathbf{T}_B \cdot {}^B\mathbf{r}_C'$$

방위의 다른 표현

■ 방위의 표현

회전행렬 (3×3 행렬) $\rightarrow$ 다른 방위를 지정할 때 마다 9 개의 원소를 지정해 주어야 함.

작은 수의 입력으로 방위를 표시할 수 있다면, 보다 효과적

■ 오일러 각 (Euler angle)과 롤-피치-요(Roll-Pitch-Yaw) 방법

물체의 방위를 연속된 세번의 회전만으로 표현할 수 있어 회전행렬 보다 효율적으로 방위를 표현하는 방법

오일러각 : 초기 방위를 나타내는 좌표계를 이동좌표계의 주축에 대해 연속적으로 세번 회전을 수행하여 특정 방위를 얻어내는 방법

롤 - 피치 - 요 방법 : 고정기준좌표계에 대해 연속적으로 세번 회전을하여 원하는 방위를 표시하는 방법

초기 좌표를 세번만 회전하므로 매 회전시의 회전량 3 개만 정의하면 되므로 9 개를 입력해야 하는 회전행렬에 비해 효율적이다.

머니플레이터 말단부를 3 자 유도로 구성함으로써 모든 방위를 구현할 수 있게 되는 근거

각각 12 가지 표현법 존재

■ Z-Y-Z 오일러 각

좌표계 {A}의 Z축에 대해 α 만큼 회전 후

이동 좌표계 즉 회전 후 얻어진 새로운 좌표계 {B'}의 Y축에 대해 β 만큼 회전한 후,

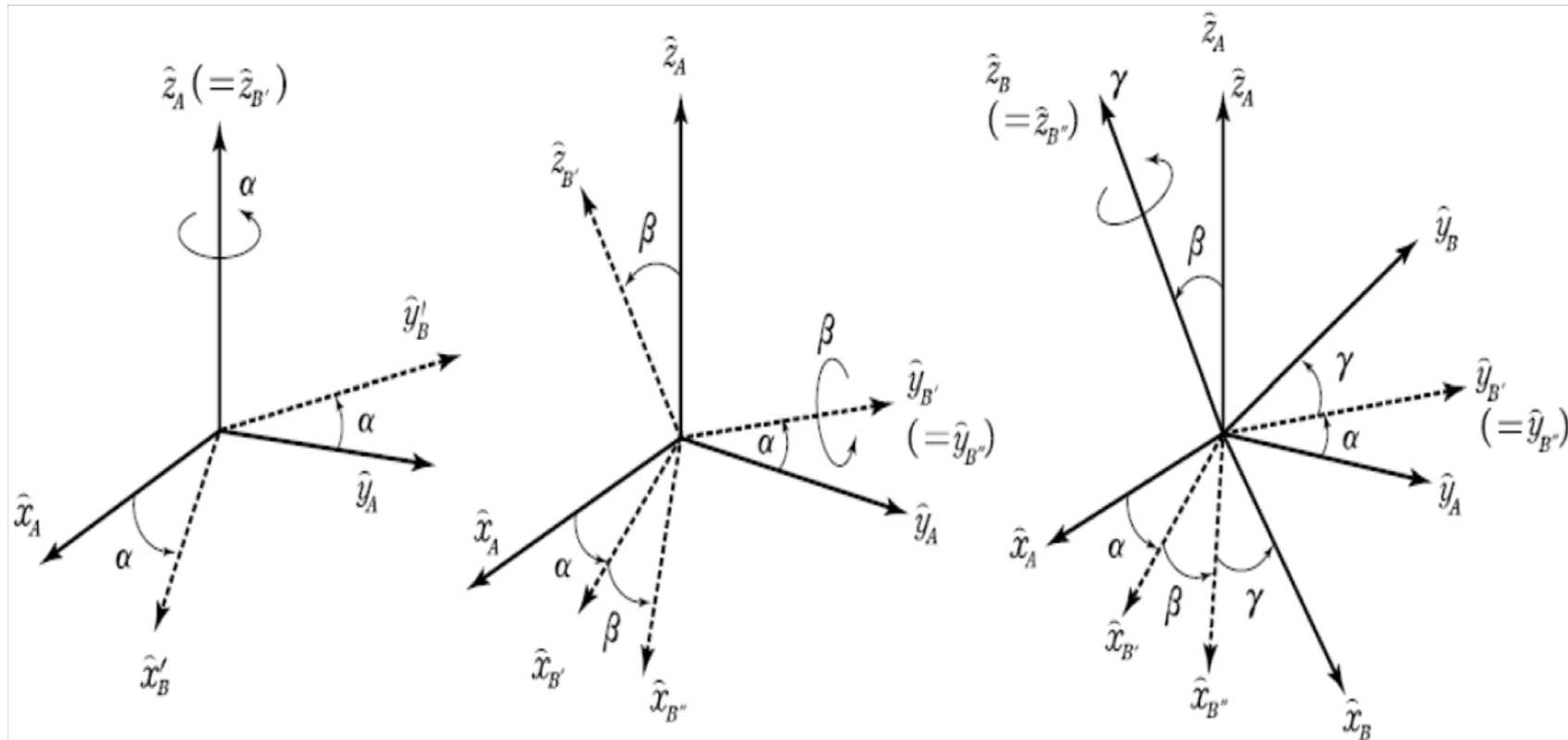
다시 새롭게 얻어진 좌표계 (이동좌표계) {B''}의 Z축에 대해 γ 만큼 연속적으로 회전을 하여 특정방위를 나타내는 좌표계 {B}를 얻는 방법

모든 회전이 이동 좌표계에 대한 회전을 이용한 상대 변환이므로 해당 변환행렬을 **앞에서부터 순차적**으로 곱한다.

$${}^A_B T = \text{Rot}[Z, \alpha] \cdot \text{Rot}[Y, \beta] \cdot \text{Rot}[Z, \gamma]$$

$$= \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} \cos[\gamma] & -\sin[\gamma] & 0 \\ \sin[\gamma] & \cos[\gamma] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos[\alpha] \cos[\beta] \cos[\gamma] - \sin[\alpha] \sin[\gamma] & -\cos[\gamma] \sin[\alpha] - \cos[\alpha] \cos[\beta] \sin[\gamma] & \cos[\alpha] \sin[\beta] \\ \cos[\beta] \cos[\gamma] \sin[\alpha] + \cos[\alpha] \sin[\gamma] & \cos[\alpha] \cos[\gamma] - \cos[\beta] \sin[\alpha] \sin[\gamma] & \sin[\alpha] \sin[\beta] \\ -\cos[\gamma] \sin[\beta] & \sin[\beta] \sin[\gamma] & \cos[\beta] \end{pmatrix}$$



■ Z-Y-X 오일러 각

좌표계 {A}의 Z축에 대해 α 만큼 회전 후

이동좌표계 즉 회전후 얻어진 새로운 좌표계 {B'}의 Y축에 대해 β 만큼 회전한 후,

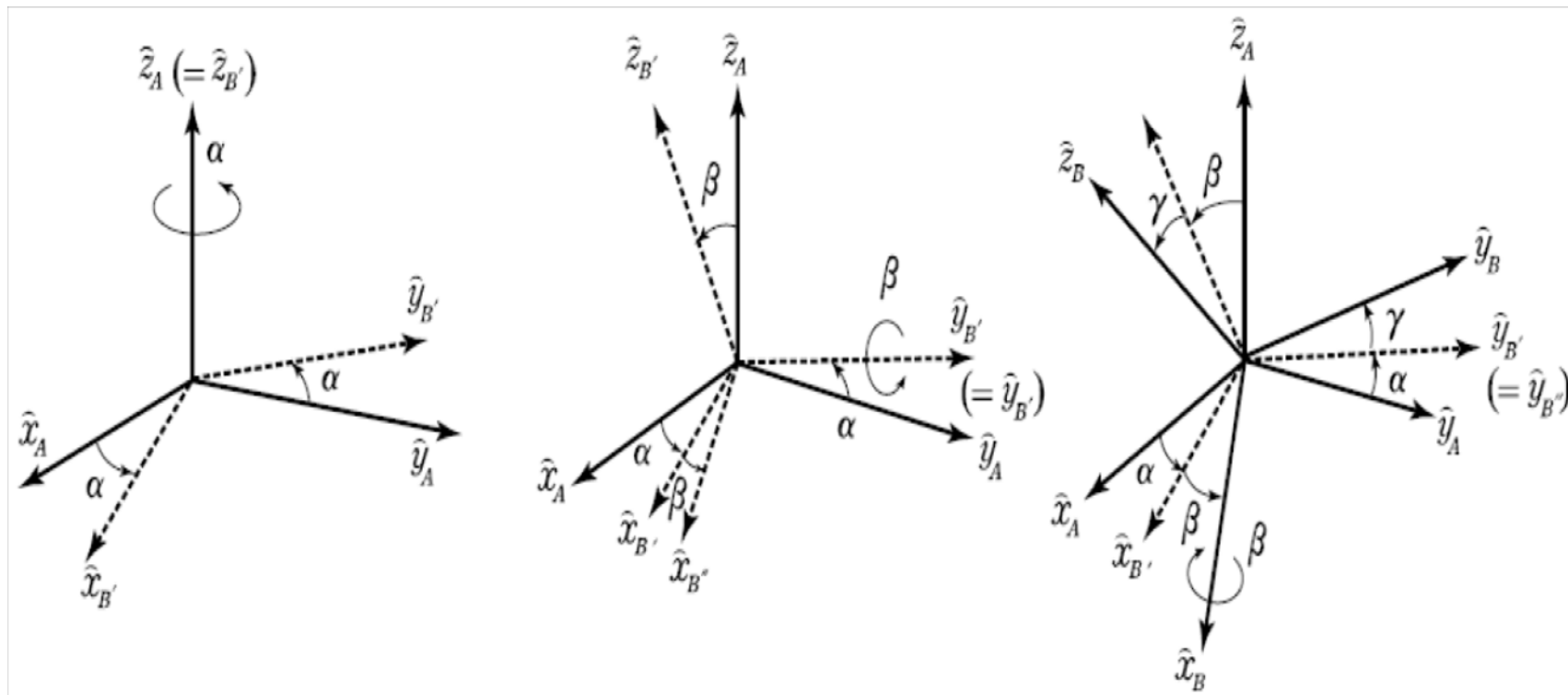
다시 새롭게 얻어진 좌표계 (이동좌표계) {B''}의 X축에 대해 γ 만큼 연속적으로 회전을 하여 특정방위를 나타내는 좌표계 {B}를 얻는 방법

모든 회전이 이동 좌표계에 대한 회전을 이용한 상대변환이므로 해당 변환행렬을 **앞에서부터 순차적**으로 곱한다.

$${}^A_B \mathbf{T} = \text{Rot}[\mathbf{z}, \alpha] \cdot \text{Rot}[\mathbf{y}, \beta] \cdot \text{Rot}[\mathbf{x}, \gamma]$$

$$= \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\gamma] & -\sin[\gamma] \\ 0 & \sin[\gamma] & \cos[\gamma] \end{pmatrix}$$

$$= \begin{pmatrix} \cos[\alpha] \cos[\beta] & -\cos[\gamma] \sin[\alpha] + \cos[\alpha] \sin[\beta] \sin[\gamma] & \cos[\alpha] \cos[\gamma] \sin[\beta] + \sin[\alpha] \sin[\gamma] \\ \cos[\beta] \sin[\alpha] & \cos[\alpha] \cos[\gamma] + \sin[\alpha] \sin[\beta] \sin[\gamma] & \cos[\gamma] \sin[\alpha] \sin[\beta] - \cos[\alpha] \sin[\gamma] \\ -\sin[\beta] & \cos[\beta] \sin[\gamma] & \cos[\beta] \cos[\gamma] \end{pmatrix}$$



■ X-Y-Z (롤-피치-요)

고정기준좌표계 {A} 의 X축에 대해 γ 만큼 회전 후

고정좌표계의 Y축에 대해 β 만큼 회전한 후,

고정좌표계의 Z축에 대해 α 만큼 연속적으로 회전을 하여 특정방위를 나타내는 좌표계 {B} 를 얻는 방법

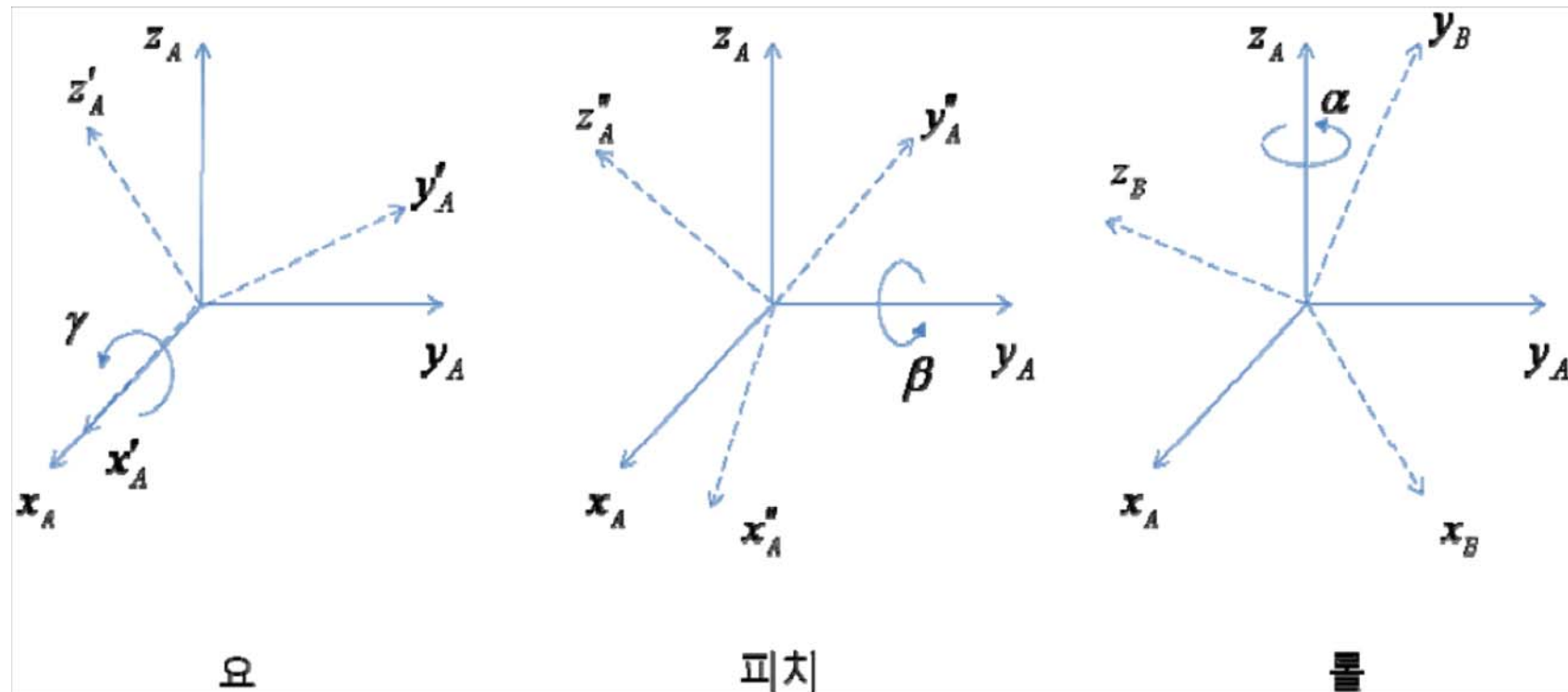
모든 회전이 고정 기준 좌표계에 대한 회전을 이용한 절대변환이므로 해당 변환행렬을 **뒤에서부터 앞으로 역순**으로 곱한다.

Z - Y - X 오일러와 동일

$${}^A_B T = \text{Rot}[Z, \alpha] \cdot \text{Rot}[Y, \beta] \cdot \text{Rot}[X, \gamma]$$

$$= \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\gamma] & -\sin[\gamma] \\ 0 & \sin[\gamma] & \cos[\gamma] \end{pmatrix}$$

$$= \begin{pmatrix} \cos[\alpha] \cos[\beta] & -\cos[\gamma] \sin[\alpha] + \cos[\alpha] \sin[\beta] \sin[\gamma] & \cos[\alpha] \cos[\gamma] \sin[\beta] + \sin[\alpha] \sin[\gamma] \\ \cos[\beta] \sin[\alpha] & \cos[\alpha] \cos[\gamma] + \sin[\alpha] \sin[\beta] \sin[\gamma] & \cos[\gamma] \sin[\alpha] \sin[\beta] - \cos[\alpha] \sin[\gamma] \\ -\sin[\beta] & \cos[\beta] \sin[\gamma] & \cos[\beta] \cos[\gamma] \end{pmatrix}$$



Rotation Matrix

$$R_x[\phi_-] := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$R_y[\theta_-] := \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} // \text{MatrixForm}$$

$$\mathbf{R}_z[\psi] := \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} // \text{MatrixForm}$$

■ Euler Angle

$$\mathbf{R} = \mathbf{R}_z[\psi] \cdot \mathbf{R}_y[\theta] \cdot \mathbf{R}_x[\phi] = \begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\mathbf{R}_z[\psi] \cdot \mathbf{R}_y[\theta] \cdot \mathbf{R}_x[\phi]$$

$$\begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\begin{pmatrix} \cos[\psi] & \sin[\psi] & 0 \\ -\sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & -\sin[\theta] \\ 0 & 1 & 0 \\ \sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & \sin[\phi] \\ 0 & -\sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] + \cos[\phi] \sin[\psi] & -\cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ -\cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi] & \cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ \sin[\theta] & -\cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

Roll: ϕ

Pitch: θ

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} + g \begin{pmatrix} \sin[\theta] \\ -\cos[\theta] \sin[\phi] \\ -\cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] & \sin[\theta] \sin[\phi] & \cos[\phi] \sin[\theta] \\ 0 & \cos[\phi] & -\sin[\phi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$(0 \ 0 \ -g) \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$(g \sin[\theta] \ -g \cos[\theta] \sin[\phi] \ -g \cos[\theta] \cos[\phi])$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[-\phi] & -\sin[-\phi] \\ 0 & \sin[-\phi] & \cos[-\phi] \end{pmatrix} \cdot \begin{pmatrix} \cos[-\theta] & 0 & \sin[-\theta] \\ 0 & 1 & 0 \\ -\sin[-\theta] & 0 & \cos[-\theta] \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} g \sin[\theta] \\ -g \cos[\theta] \sin[\phi] \\ -g \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[-\phi] & -\sin[-\phi] \\ 0 & \sin[-\phi] & \cos[-\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & \sin[\phi] \\ 0 & -\sin[\phi] & \cos[\phi] \end{pmatrix}$$

yaw: φ

$$\mathbf{x} = \begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix}$$

$$\begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} 1 & \sin[\phi] \tan[\theta] & \cos[\phi] \tan[\theta] \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] \sec[\theta] & \cos[\phi] \sec[\theta] \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\tan[\theta] = \frac{\sin[\theta]}{\cos[\theta]}$$

$$\sec[\theta] = \frac{1}{\cos[\theta]}$$

Euler Angle

Rotation Matrix

■ 3D orthonormal basis → unit vectors

$$\hat{u}, \hat{v}, \hat{w}$$

$$A = \begin{pmatrix} \hat{u}_x & \hat{v}_x & \hat{w}_x \\ \hat{u}_y & \hat{v}_y & \hat{w}_y \\ \hat{u}_z & \hat{v}_z & \hat{w}_z \end{pmatrix}$$

$$|\hat{u}| = |\hat{v}| = 1$$

$$\hat{u} \cdot \hat{v} = 0$$

$$\hat{u} \times \hat{v} = \hat{w}$$

- single rotation about some axis (unit vector)

- rotation vector

$$\hat{\mathbf{e}} = \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix} \Rightarrow \mathbf{v} = \theta \hat{\mathbf{e}}$$

- Quaternions

- normalized four-dimensional vector

$$\hat{\mathbf{q}} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

- Euler axis

$$\hat{\mathbf{e}} = \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix}$$

- Angle

$$\theta$$

■ vector's elements

$$q_1 = e_x \sin\left[\frac{\theta}{2}\right]$$

$$q_2 = e_y \sin\left[\frac{\theta}{2}\right]$$

$$q_3 = e_z \sin\left[\frac{\theta}{2}\right]$$

$$q_4 = \cos\left[\frac{\theta}{2}\right]$$

■ the quaternion parametrization obeys

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

■ hypercomplex numbers {i,j,k}

$$a + b i + c j + d k$$

$$i^2 = j^2 = k^2 = -1$$

$$i j = -j i = k$$

$$j k = -k j = i$$

$$k i = -i k = j$$

■ Quaternion multiplication

$$\tilde{q} \otimes q = \begin{pmatrix} q_4 & -q_3 & -q_2 & -q_1 \\ -q_3 & q_4 & -q_1 & q_2 \\ -q_2 & -q_1 & q_4 & q_3 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix} \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

$$A_3 = A_2 A_1$$

$$q_3 = q_1 \otimes q_2$$

Rotation matrix ↔ Euler angles

3 - 1 - 3 Euler angles ϕ , θ , ψ (around Z, X, Z axis)

Euler angles (ϕ , θ , ψ)

$$A = \begin{pmatrix} \hat{u}_x & \hat{v}_x & \hat{w}_x \\ \hat{u}_y & \hat{v}_y & \hat{w}_y \\ \hat{u}_z & \hat{v}_z & \hat{w}_z \end{pmatrix}$$

$$\phi = \text{ArcTan}\left[\frac{A_{31}}{A_{32}}\right]$$

$$\theta = \text{ArcCos}[A_{33}]$$

$$\psi = -\text{ArcTan}\left[\frac{A_{13}}{A_{23}}\right]$$

$$A = A_3 A_2 A_1 = A_Z A_Y A_X$$

$$A_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & \sin[\phi] \\ 0 & -\sin[\phi] & \cos[\phi] \end{pmatrix}$$

$$A_Y = \begin{pmatrix} \cos[\theta] & 0 & -\sin[\theta] \\ 0 & 1 & 0 \\ \sin[\theta] & 0 & \cos[\theta] \end{pmatrix}$$

$$A_Z = \begin{pmatrix} \cos[\psi] & \sin[\psi] & 0 \\ -\sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} \cos[\psi] & \sin[\psi] & 0 \\ -\sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & -\sin[\theta] \\ 0 & 1 & 0 \\ \sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & \sin[\phi] \\ 0 & -\sin[\phi] & \cos[\phi] \end{pmatrix}$$

$$A = \begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] + \cos[\phi] \sin[\psi] & -\cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ -\cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi] & \cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ \sin[\theta] & -\cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

Rotation matrix

■ Decomposing

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{32} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

$$R = \begin{pmatrix} \cos[\theta_y] \cos[\theta_z] & \cos[\theta_z] \sin[\theta_x] \sin[\theta_y] - \cos[\theta_x] \sin[\theta_z] & \cos[\theta_x] \cos[\theta_z] \sin[\theta_y] + \sin[\theta_x] \sin[\theta_z] \\ \cos[\theta_y] \sin[\theta_z] & \cos[\theta_x] \cos[\theta_z] + \sin[\theta_x] \sin[\theta_y] \sin[\theta_z] & -\cos[\theta_z] \sin[\theta_x] + \cos[\theta_x] \sin[\theta_y] \sin[\theta_z] \\ -\sin[\theta_y] & \cos[\theta_y] \sin[\theta_x] & \cos[\theta_x] \cos[\theta_y] \end{pmatrix}$$

$$\theta_x = \text{ArcTan}\left[\frac{r_{32}}{r_{33}}\right]$$

$$\theta_y = \text{ArcTan}\left[-\frac{r_{31}}{\sqrt{r_{32}^2 + r_{33}^2}}\right]$$

$$\theta_z = \text{ArcTan}\left[\frac{r_{21}}{r_{11}}\right]$$

■ Composing

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\theta_x] & -\sin[\theta_x] \\ 0 & \sin[\theta_x] & \cos[\theta_x] \end{pmatrix}$$

$$Y = \begin{pmatrix} \cos[\theta_y] & 0 & \sin[\theta_y] \\ 0 & 1 & 0 \\ -\sin[\theta_y] & 0 & \cos[\theta_y] \end{pmatrix}$$

$$Z = \begin{pmatrix} \cos[\theta_z] & -\sin[\theta_z] & 0 \\ \sin[\theta_z] & \cos[\theta_z] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R = ZYX$$

$$R = \begin{pmatrix} \cos[\theta_z] & -\sin[\theta_z] & 0 \\ \sin[\theta_z] & \cos[\theta_z] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta_y] & 0 & \sin[\theta_y] \\ 0 & 1 & 0 \\ -\sin[\theta_y] & 0 & \cos[\theta_y] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\theta_x] & -\sin[\theta_x] \\ 0 & \sin[\theta_x] & \cos[\theta_x] \end{pmatrix}$$

$$\theta_x \rightarrow \{-\pi, \pi\}$$

$$\theta_y \rightarrow \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$$

$$\theta_z \rightarrow \{-\pi, \pi\}$$

2D Rotation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos[\theta] & -\sin[\theta] \\ \sin[\theta] & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos[\theta] - y \sin[\theta] \\ y \cos[\theta] + x \sin[\theta] \end{pmatrix}$$

Rotation matrix ↔ Euler axis/angles

$$\hat{\mathbf{e}} = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

$$\theta = \text{ArcCos} \left[\frac{1}{2} (\mathbf{A}_{11} + \mathbf{A}_{22} + \mathbf{A}_{33} - 1) \right]$$

$$\mathbf{e}_1 = \frac{\mathbf{A}_{32} - \mathbf{A}_{23}}{2 \sin[\theta]}$$

$$\mathbf{e}_2 = \frac{\mathbf{A}_{13} - \mathbf{A}_{31}}{2 \sin[\theta]}$$

$$\mathbf{e}_3 = \frac{\mathbf{A}_{21} - \mathbf{A}_{12}}{2 \sin[\theta]}$$

General rotations

$$\mathbf{R}_X[\gamma] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\gamma] & -\sin[\gamma] \\ 0 & \sin[\gamma] & \cos[\gamma] \end{pmatrix} : \text{roll } \gamma$$

$$\mathbf{R}_Y[\beta] = \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} : \text{pitch } \beta$$

$$\mathbf{R}_Z[\alpha] = \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} : \text{yaw } \alpha$$

$$\mathbf{R} = \mathbf{R}_X[\gamma] \mathbf{R}_Y[\beta] \mathbf{R}_Z[\alpha]$$

Euler Angle

$$\mathbf{R} = \mathbf{R}_Z[\gamma] \mathbf{R}_X[\beta] \mathbf{R}_Y[\alpha]$$

Rotation matrix from axis and angle

■ Euler axis

$$\mathbf{e} = \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}$$

■ Angle

$$\theta$$

■ Rotation

$$\mathbf{R} = \mathbf{I} \cos[\theta] + \sin[\theta] [\mathbf{u}]_{\times} + (1 - \cos[\theta]) \mathbf{u} \otimes \mathbf{u}$$

$$\mathbf{u} \otimes \mathbf{u} = \begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{pmatrix}$$

$$[\mathbf{u}]_{\times} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_x u_y (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_x u_z (1 - \cos[\theta]) - u_y \sin[\theta] & u_y u_z (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix}$$

Rotation matrix from axis and angle

$$\mathbf{R}[\theta_] := \begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_y u_x (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_z u_x (1 - \cos[\theta]) - u_y \sin[\theta] & u_z u_y (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix}$$

■ concisely

$$\mathbf{R} = \mathbf{I} \cdot \cos[\theta] + \sin[\theta] \mathbf{u}_x + (1 - \cos[\theta]) \mathbf{u} \otimes \mathbf{u}$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{u}_x = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}$$

$$\mathbf{u} \otimes \mathbf{u} = \begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{pmatrix}$$

■ ex

$$\{u_x, u_y, u_z\} = \{0, 0, 1\}; \begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_y u_x (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_z u_x (1 - \cos[\theta]) - u_y \sin[\theta] & u_z u_y (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix} \cdot \begin{pmatrix} Ax \\ Ay \\ Az \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} Ax \cos[\theta] - Ay \sin[\theta] \\ Ay \cos[\theta] + Ax \sin[\theta] \\ Az \end{pmatrix}$$

$$\begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_y u_x (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_z u_x (1 - \cos[\theta]) - u_y \sin[\theta] & u_z u_y (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix} \cdot \begin{pmatrix} Ax \\ Ay \\ Az \end{pmatrix} // \text{FullSimplify} // \text{InputForm}$$

```
{Ax*cos[θ] + Az*sin[θ]*Subscript[u, y] - (-1 + Cos[θ])*Subscript[u, x]*
  (Ax*Subscript[u, x] + Ay*Subscript[u, y]) -
  (Ay*sin[θ] + Az*(-1 + Cos[θ])*Subscript[u, x])*Subscript[u, z]},
{Ay*cos[θ] - Subscript[u, x]*(Az*sin[θ] + Ax*(-1 + Cos[θ])*Subscript[u, y]) +
  Ax*sin[θ]*Subscript[u, z] - (-1 + Cos[θ])*Subscript[u, y]*
  (Ay*Subscript[u, y] + Az*Subscript[u, z])},
{Az*cos[θ] + Ay*sin[θ]*Subscript[u, x] - Ax*sin[θ]*Subscript[u, y] -
  (-1 + Cos[θ])*(Ax*Subscript[u, x] + Ay*Subscript[u, y])*Subscript[u, z] -
  Az*(-1 + Cos[θ])*Subscript[u, z]^2}
```

$R_x[\phi_] := // \text{MatrixForm}$

$$R_y[\theta_] := \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} // \text{MatrixForm}$$

$$R_z[\psi_] := \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[y] & -\sin[y] & 0 \\ \sin[y] & \cos[y] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[p] & 0 & \sin[p] \\ 0 & 1 & 0 \\ -\sin[p] & 0 & \cos[p] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[r] & -\sin[r] \\ 0 & \sin[r] & \cos[r] \end{pmatrix} \cdot \begin{pmatrix} Wx \\ Wy \\ Wz \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} Wx \cos[p] \cos[y] + Wy (\cos[y] \sin[p] \sin[r] - \cos[r] \sin[y]) + Wz (\cos[r] \cos[y] \sin[p] + \sin[r] \sin[y]) \\ Wx \cos[p] \sin[y] + Wz (-\cos[y] \sin[r] + \cos[r] \sin[p] \sin[y]) + Wy (\cos[r] \cos[y] + \sin[p] \sin[r] \sin[y]) \\ Wz \cos[p] \cos[r] - Wx \sin[p] + Wy \cos[p] \sin[r] \end{pmatrix}$$

$$\begin{pmatrix} \cos[p] & 0 & \sin[p] \\ 0 & 1 & 0 \\ -\sin[p] & 0 & \cos[p] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[r] & -\sin[r] \\ 0 & \sin[r] & \cos[r] \end{pmatrix} \cdot \begin{pmatrix} Vx \\ Vy \\ Vz \end{pmatrix}$$

$$\{\{Vx \cos[p] + Vz \cos[r] \sin[p] + Vy \sin[p] \sin[r]\}, \{Vy \cos[r] - Vz \sin[r]\}, \{Vz \cos[p] \cos[r] - Vx \sin[p] + Vy \cos[p] \sin[r]\}\}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[r] & -\sin[r] \\ 0 & \sin[r] & \cos[r] \end{pmatrix} \cdot \begin{pmatrix} Ux \\ Uy \\ Uz \end{pmatrix}$$

$$\{\{Ux\}, \{Uy \cos[r] - Uz \sin[r]\}, \{Uz \cos[r] + Uy \sin[r]\}\}$$

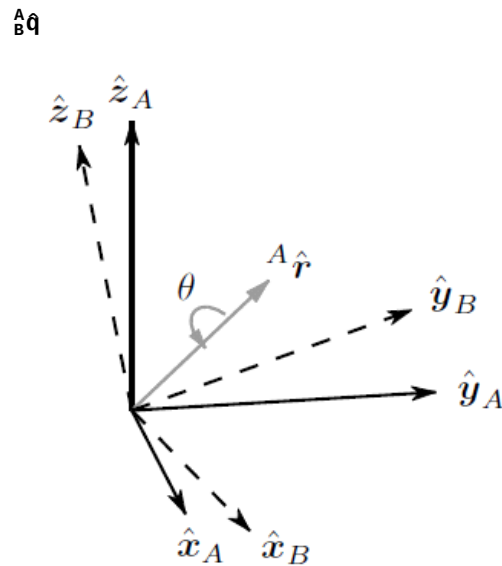
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[-r] & -\sin[-r] \\ 0 & \sin[-r] & \cos[-r] \end{pmatrix} \cdot \begin{pmatrix} \cos[-p] & 0 & \sin[-p] \\ 0 & 1 & 0 \\ -\sin[-p] & 0 & \cos[-p] \end{pmatrix} \cdot \begin{pmatrix} \cos[-y] & -\sin[-y] & 0 \\ \sin[-y] & \cos[-y] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} Xx \\ Xy \\ Xz \end{pmatrix}$$

$$\{\{Xx \cos[p] \cos[y] - Xz \sin[p] + Xy \cos[p] \sin[y]\}, \\ \{Xz \cos[p] \sin[r] + Xx (\cos[y] \sin[p] \sin[r] - \cos[r] \sin[y]) + Xy (\cos[r] \cos[y] + \sin[p] \sin[r] \sin[y])\}, \\ \{Xz \cos[p] \cos[r] + Xy (-\cos[y] \sin[r] + \cos[r] \sin[p] \sin[y]) + Xx (\cos[r] \cos[y] \sin[p] + \sin[r] \sin[y])\}\}$$

An efficient orientation filter for inertial and inertial/magnetic sensor arrays

Quaternion representation

- A quaternion is a four-dimensional complex number
- An arbitrary orientation of frame B relative to frame A



- A rotation of angle θ around an axis ${}^A\hat{r}$ defined in frame A

$${}^A\hat{r} = (r_x \ r_y \ r_z)$$

$${}^B\hat{q} = (q_1 \ q_2 \ q_3 \ q_4) = \left(\cos\left[\frac{\theta}{2}\right] \ -r_x \sin\left[\frac{\theta}{2}\right] \ -r_y \sin\left[\frac{\theta}{2}\right] \ -r_z \sin\left[\frac{\theta}{2}\right] \right)$$

- A quaternion conjugate

$${}^B\hat{q}^* = {}^B\hat{q} = (q_1 \ -q_2 \ -q_3 \ -q_4)$$

- A quaternion product $\otimes$

$${}^A\hat{q} = {}^B\hat{q} \otimes {}^A\hat{q}$$

- Hamilton rule

$$\begin{aligned} \mathbf{a} \otimes \mathbf{b} &= (a_1 \ a_2 \ a_3 \ a_4) \otimes (b_1 \ b_2 \ b_3 \ b_4) \\ &= \begin{pmatrix} a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4 \\ a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3 \\ a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2 \\ a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1 \end{pmatrix}^T \end{aligned}$$

- A three dimensional vector

$${}^B\mathcal{U} = {}^A\hat{q} \otimes {}^A\mathcal{U} \otimes {}^B\hat{q}^*$$

- The orientation ${}^B\hat{q} \rightarrow$ The rotation matrix

$${}^B\mathbf{R} = \begin{pmatrix} 2q_1^2 - 1 + 2q_2^2 & 2(q_2q_3 + q_1q_4) & 2(q_2q_4 - q_1q_3) \\ 2(q_2q_3 - q_1q_4) & 2q_1^2 - 1 + 2q_3^2 & 2(q_3q_4 + q_1q_2) \\ 2(q_2q_4 + q_1q_3) & 2(q_3q_4 - q_1q_2) & 2q_1^2 - 1 + 2q_4^2 \end{pmatrix}$$

■ Euler Angle

ϕ around $\hat{x}_B$, θ around $\hat{y}_B$, ψ around $\hat{z}_B$

$$\phi = \text{ArcTan}[\hat{A}_B R_{33}, \hat{A}_B R_{32}] = \text{ArcTan}[2 q_1^2 - 1 + 2 q_4^2, 2 (q_3 q_4 - q_1 q_2)]$$

$$\theta = -\text{ArcTan}[\sqrt{1 - \hat{A}_B R_{31}^2}, \hat{A}_B R_{31}] = -\text{ArcSin}[2 (q_2 q_4 + q_1 q_3)]$$

$$\psi = \text{ArcTan}[\hat{A}_B R_{11}, \hat{A}_B R_{12}] = \text{ArcTan}[2 q_1^2 - 1 + 2 q_2^2, 2 (q_2 q_3 - q_1 q_4)]$$

Filter derivation

Orientation from angular rate

■ A tri-axis gyroscope

■ The angular rate

$$^S \omega = (\theta \quad \omega_x \quad \omega_y \quad \omega_z)$$

■ The quaternion derivative

$$\dot{\hat{q}} = \frac{1}{2} \hat{q} \otimes ^S \omega$$

- The quaternion derivative measured at time t

$$\dot{\hat{\mathbf{q}}}_{\omega, t} = \frac{1}{2} \hat{\mathbf{q}}_{\text{est}, t-1} \otimes \omega_t$$

- The orientation of the earth frame at time t

$$\hat{\mathbf{q}}_{\omega, t} = \hat{\mathbf{q}}_{\text{est}, t-1} + \dot{\hat{\mathbf{q}}}_{\omega, t} \Delta t$$

The sampling period Δt

Orientation from vector observations

- A tri-axis accelerometer (Linear accelerations due to motion)

the field of gravity

- A tri-axis magnetometer (Local magnetic flux and distortions)

the earth's magnetic field

- A quaternion

$$\hat{\mathbf{q}} = (q_1 \quad q_2 \quad q_3 \quad q_4)$$

- A predefined reference direction

$$\mathbf{d} = (0 \quad d_x \quad d_y \quad d_z)$$

- The measured direction

$$^S\mathbf{s} = (0 \quad s_x \quad s_y \quad s_z)$$

- The objective function

$$f[\mathbf{\hat{q}}, \mathbf{d}, \mathbf{s}] = \mathbf{\hat{q}}^* \otimes \mathbf{d} \otimes \mathbf{\hat{q}} - \mathbf{s}$$

- A quaternion may be found

$$\min_{\mathbf{\hat{q}} \in \mathbb{R}^4} f[\mathbf{\hat{q}}, \mathbf{d}, \mathbf{s}]$$

The gradient descent algorithm

- An orientation estimation of $\mathbf{\hat{q}}_{n+1}$

$$\mathbf{\hat{q}}_{k+1} = \mathbf{\hat{q}}_k - \mu \frac{\nabla f[\mathbf{\hat{q}}, \mathbf{d}, \mathbf{s}]}{|\nabla f[\mathbf{\hat{q}}, \mathbf{d}, \mathbf{s}]|}, \quad k = 0, 1, \dots, n$$

A step-size μ

- The gradient of the solution surface (General form)

$$\nabla f[\mathbf{\hat{q}}_k, \mathbf{d}, \mathbf{s}] = \mathbf{J}^T[\mathbf{\hat{q}}_k, \mathbf{d}] f[\mathbf{\hat{q}}_k, \mathbf{d}, \mathbf{s}]$$

■ A objective function

$$f[\mathbf{s}_{\hat{\mathbf{E}}\hat{\mathbf{q}}_k}, \mathbf{E}\hat{\mathbf{d}}, \mathbf{s}_{\hat{\mathbf{s}}}] = \begin{pmatrix} 2 d_x (\frac{1}{2} - q_3^2 - q_4^2) + 2 d_y (q_1 q_4 + q_2 q_3) + 2 d_z (q_2 q_4 - q_1 q_3) - s_x \\ 2 d_x (q_2 q_3 - q_1 q_4) + 2 d_y (\frac{1}{2} - q_2^2 - q_4^2) + 2 d_z (q_1 q_2 + q_3 q_4) - s_y \\ 2 d_x (q_1 q_3 + q_2 q_4) + 2 d_y (q_3 q_4 - q_1 q_2) + 2 d_z (\frac{1}{2} - q_2^2 - q_3^2) - s_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}[\mathbf{s}_{\hat{\mathbf{E}}\hat{\mathbf{q}}_k}, \mathbf{E}\hat{\mathbf{d}}] = \begin{pmatrix} 2 d_y q_4 - 2 d_z q_3 & 2 d_y q_3 + 2 d_z q_4 & -4 d_x q_3 + 2 d_y q_2 - 2 d_z q_1 & -4 d_x q_4 + 2 d_y q_1 + 2 d_z q_2 \\ -2 d_x q_4 + 2 d_z q_2 & 2 d_x q_3 - 4 d_y q_2 + 2 d_z q_1 & 2 d_x q_2 + 2 d_z q_4 & -2 d_x q_1 - 4 d_y q_4 + 2 d_z q_3 \\ 2 d_x q_3 - 2 d_y q_2 & 2 d_x q_4 - 2 d_y q_1 - 4 d_z q_2 & 2 d_x q_1 + 2 d_y q_4 - 4 d_z q_3 & 2 d_x q_2 + 2 d_y q_3 \end{pmatrix}$$

The direction of gravity (the vertical axis, z axis)

■ An appropriate convention (the equations simplify)

$$\mathbf{E}\hat{\mathbf{g}} = (0 \quad 0 \quad 0 \quad 1)$$

■ Normalized accelerometer measurement

$$\mathbf{s}\hat{\mathbf{a}} = (0 \quad a_x \quad a_y \quad a_z)$$

■ A objective function

$$\mathbf{f}_g[\mathbf{s}_{\hat{\mathbf{q}}}, \mathbf{s}_{\hat{\mathbf{a}}}] = \begin{pmatrix} 2 (q_2 q_4 - q_1 q_3) - a_x \\ 2 (q_1 q_2 + q_3 q_4) - a_y \\ 2 (\frac{1}{2} - q_2^2 - q_3^2) - a_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}_g[\mathbf{s}_{\hat{\mathbf{q}}}] = \begin{pmatrix} -2 q_3 & 2 q_4 & -2 q_1 & 2 q_2 \\ 2 q_2 & 2 q_1 & 2 q_4 & 2 q_3 \\ 0 & -4 q_2 & -4 q_3 & 0 \end{pmatrix}$$

The earth's magnetic field (one horizontal axis, the vertical axis)

■ An appropriate convention (the equations simplify)

$$\mathbf{e}_{\hat{\mathbf{b}}} = (0 \quad b_x \quad 0 \quad b_z)$$

■ Normalized accelerometer measurement

$$\mathbf{s}_{\hat{\mathbf{m}}} = (0 \quad m_x \quad m_y \quad m_z)$$

■ A objective function

$$\mathbf{f}_b[\mathbf{s}_{\hat{\mathbf{q}}}, \mathbf{e}_{\hat{\mathbf{b}}}, \mathbf{s}_{\hat{\mathbf{m}}}] = \begin{pmatrix} 2 b_x (\frac{1}{2} - q_3^2 - q_4^2) + 2 b_z (q_2 q_4 - q_1 q_3) - m_x \\ 2 b_x (q_2 q_3 - q_1 q_4) + 2 b_z (q_1 q_2 + q_3 q_4) - m_y \\ 2 b_x (q_1 q_3 + q_2 q_4) + 2 b_z (\frac{1}{2} - q_2^2 - q_3^2) - m_z \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}_b \left[\begin{smallmatrix} \mathbf{s} \\ \mathbf{q} \end{smallmatrix}, \begin{smallmatrix} \mathbf{e} \\ \mathbf{b} \end{smallmatrix} \right] = \begin{pmatrix} -2 \mathbf{b}_z \mathbf{q}_3 & 2 \mathbf{b}_z \mathbf{q}_4 & -4 \mathbf{b}_x \mathbf{q}_3 - 2 \mathbf{b}_z \mathbf{q}_1 & -4 \mathbf{b}_x \mathbf{q}_4 + 2 \mathbf{b}_z \mathbf{q}_2 \\ -2 \mathbf{b}_x \mathbf{q}_4 + 2 \mathbf{b}_z \mathbf{q}_2 & 2 \mathbf{b}_x \mathbf{q}_3 + 2 \mathbf{b}_z \mathbf{q}_1 & 2 \mathbf{b}_x \mathbf{q}_2 + 2 \mathbf{b}_z \mathbf{q}_4 & -2 \mathbf{b}_x \mathbf{q}_1 + 2 \mathbf{b}_z \mathbf{q}_3 \\ 2 \mathbf{b}_x \mathbf{q}_3 & 2 \mathbf{b}_x \mathbf{q}_4 - 4 \mathbf{b}_z \mathbf{q}_2 & 2 \mathbf{b}_x \mathbf{q}_1 - 4 \mathbf{b}_z \mathbf{q}_3 & 2 \mathbf{b}_x \mathbf{q}_2 \end{pmatrix}$$

The solution surface → minimum

■ The objective function

$$\mathbf{f}_{g,b} \left[\begin{smallmatrix} \mathbf{s} \\ \mathbf{q} \end{smallmatrix}, \begin{smallmatrix} \mathbf{s} \\ \mathbf{a} \end{smallmatrix}, \begin{smallmatrix} \mathbf{e} \\ \mathbf{b} \end{smallmatrix}, \begin{smallmatrix} \mathbf{s} \\ \mathbf{m} \end{smallmatrix} \right] = \begin{pmatrix} \mathbf{f}_g \left[\begin{smallmatrix} \mathbf{s} \\ \mathbf{q} \end{smallmatrix}, \begin{smallmatrix} \mathbf{s} \\ \mathbf{a} \end{smallmatrix} \right] \\ \mathbf{f}_b \left[\begin{smallmatrix} \mathbf{s} \\ \mathbf{q} \end{smallmatrix}, \begin{smallmatrix} \mathbf{e} \\ \mathbf{b} \end{smallmatrix}, \begin{smallmatrix} \mathbf{s} \\ \mathbf{m} \end{smallmatrix} \right] \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}_{g,b} \left[\begin{smallmatrix} \mathbf{s} \\ \mathbf{q} \end{smallmatrix}, \begin{smallmatrix} \mathbf{e} \\ \mathbf{b} \end{smallmatrix} \right] = \begin{pmatrix} \mathbf{J}_g^T \left[\begin{smallmatrix} \mathbf{s} \\ \mathbf{q} \end{smallmatrix} \right] \\ \mathbf{J}_b^T \left[\begin{smallmatrix} \mathbf{s} \\ \mathbf{q} \end{smallmatrix}, \begin{smallmatrix} \mathbf{e} \\ \mathbf{b} \end{smallmatrix} \right] \end{pmatrix}$$

The estimate orientation $\begin{smallmatrix} \mathbf{s} \\ \mathbf{q} \end{smallmatrix}_{\nabla, t}$

$$\begin{smallmatrix} \mathbf{s} \\ \mathbf{q} \end{smallmatrix}_{\nabla, t} = \begin{smallmatrix} \mathbf{s} \\ \mathbf{q} \end{smallmatrix}_{\text{est}, t} - \mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|}$$

■ The convergence rate governed by $\mu_t \geq$ the physical rate

A previous estimate of orientation $\begin{smallmatrix} \mathbf{s} \\ \mathbf{q} \end{smallmatrix}_{\text{est}, t-1}$

■ The objective function gradient

$$\nabla \mathbf{f} = \begin{Bmatrix} \mathbf{J}_g^T [\hat{\mathbf{q}}_{\text{est}, t-1}] \mathbf{f}_g [\hat{\mathbf{q}}_{\text{est}, t-1}, \mathbf{s}_a] \\ \mathbf{J}_{g,b}^T [\hat{\mathbf{q}}_{\text{est}, t-1}, \mathbf{E}_b] \mathbf{f}_{g,b} [\hat{\mathbf{q}}_{\text{est}, t-1}, \mathbf{s}_a, \mathbf{E}_b, \mathbf{s}_m] \end{Bmatrix}$$

The convergence rate of $\hat{\mathbf{q}}_{\nabla, t}$

The physical orientation rate $\hat{\mathbf{q}}_{\omega, t}$

■ An optimal value of the step-size

$$\mu_t = \alpha \mid \hat{\mathbf{q}}_{\omega, t} \mid \Delta t, \alpha > 1$$

Filter fusion algorithm

■ An estimated orientation

$$\hat{\mathbf{q}}_{\text{est}, t} = \gamma_t \hat{\mathbf{q}}_{\nabla, t} + (1 - \gamma_t) \hat{\mathbf{q}}_{\omega, t}, \quad 0 \leq \gamma_t \leq 1$$

■ $\beta, \frac{\mu_t}{\Delta t}$

$$\text{The weighted divergence of } \hat{\mathbf{q}}_{\omega} \underset{\beta}{=} \text{The weighted convergence of } \hat{\mathbf{q}}_{\nabla} \underset{\frac{\mu_t}{\Delta t}}{=}$$

■ The magnitude of a quaternion derivative == The gyroscope measurement error

$$(1 - \gamma_t) \beta = \gamma_t \frac{\mu_t}{\Delta t}$$

$$\gamma_t = \frac{\beta}{\frac{\mu_t}{\Delta t} + \beta}$$

■ The optimal fusion

- the convergence rate of $\hat{\mathbf{q}}_{\nabla} \geq \alpha \rightarrow$ the physical rate of change of orientation

$$\hat{\mathbf{q}}_{\nabla, t} \approx -\mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|}$$

$$\gamma_t \approx \frac{\beta \Delta t}{\mu_t}$$

■ Simplify

- An estimated orientation

$$\hat{\mathbf{q}}_{\text{est}, t} = \frac{\beta \Delta t}{\mu_t} \left(-\mu_t \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) + (1 - \theta) (\hat{\mathbf{q}}_{\text{est}, t-1} + \hat{\mathbf{q}}_{\omega, t} \Delta t)$$

$$\hat{\mathbf{q}}_{\text{est}, t} = -\beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \Delta t + \hat{\mathbf{q}}_{\text{est}, t-1} + \hat{\mathbf{q}}_{\omega, t} \Delta t$$

$$\hat{\mathbf{q}}_{\text{est}, t} = \hat{\mathbf{q}}_{\text{est}, t-1} + \hat{\mathbf{q}}_{\omega, t} \Delta t - \beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \Delta t$$

$$\hat{\mathbf{q}}_{\text{est}, t} = \hat{\mathbf{q}}_{\text{est}, t-1} + \left(\hat{\mathbf{q}}_{\omega, t} - \beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

$$\hat{\mathbf{q}}_{\text{est}, t} = \hat{\mathbf{q}}_{\text{est}, t-1} + \left(\hat{\mathbf{q}}_{\omega, t} - \beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

$$\hat{\mathbf{q}}_{\text{est}, t} = \hat{\mathbf{q}}_{\text{est}, t-1} + \left(\hat{\mathbf{q}}_{\omega, t} - \beta \hat{\mathbf{q}}_{\epsilon, t} \right) \Delta t$$

$$\hat{\mathbf{q}}_{\text{est},t} = \hat{\mathbf{q}}_{\text{est},t-1} + \hat{\dot{\mathbf{q}}}_{\text{est},t} \Delta t$$

$$\hat{\dot{\mathbf{q}}}_{\text{est},t} = \hat{\dot{\mathbf{q}}}_{\text{est},t-1} + \left(\frac{1}{2} \hat{\mathbf{q}}_{\text{est},t-1} \otimes \mathbf{s}_{\omega_t} - \beta \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) \Delta t$$

The estimated orientation rate $\hat{\dot{\mathbf{q}}}_{\text{est}}$

The rate of change of orientation measured by the gyroscopes $\hat{\dot{\mathbf{q}}}_{\omega}$

The direction of the estimated error $\hat{\dot{\mathbf{q}}}_{\epsilon}$

■ Simplified to equation

$$\hat{\mathbf{q}}_{\text{est},t} = \hat{\mathbf{q}}_{\text{est},t-1} + \hat{\dot{\mathbf{q}}}_{\text{est},t} \Delta t$$

■ The estimated rate of change of orientation

$$\hat{\dot{\mathbf{q}}}_{\text{est},t} = \hat{\dot{\mathbf{q}}}_{\omega,t} - \beta \hat{\dot{\mathbf{q}}}_{\epsilon,t}$$

■ The quaternion derivative measured at time t

$$\hat{\dot{\mathbf{q}}}_{\omega,t} = \frac{1}{2} \hat{\mathbf{q}}_{\text{est},t-1} \otimes \mathbf{s}_{\omega_t}$$

■ The direction of the error of $\hat{\dot{\mathbf{q}}}_{\text{est},t}$

$$\hat{\dot{\mathbf{q}}}_{\epsilon,t} = \frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|}$$

■ The objective function

$$\mathbf{f}_{g,b}[\hat{\mathbf{e}}\hat{\mathbf{q}}, \mathbf{s}\hat{\mathbf{a}}, \mathbf{e}\hat{\mathbf{b}}, \mathbf{s}\hat{\mathbf{m}}] = \begin{pmatrix} \mathbf{f}_g[\hat{\mathbf{e}}\hat{\mathbf{q}}, \mathbf{s}\hat{\mathbf{a}}] \\ \mathbf{f}_b[\hat{\mathbf{e}}\hat{\mathbf{q}}, \mathbf{e}\hat{\mathbf{b}}, \mathbf{s}\hat{\mathbf{m}}] \end{pmatrix}$$

■ Jacobian

$$\mathbf{J}_{g,b}[\hat{\mathbf{e}}\hat{\mathbf{q}}, \mathbf{e}\hat{\mathbf{b}}] = \begin{pmatrix} \mathbf{J}_g^T[\hat{\mathbf{e}}\hat{\mathbf{q}}] \\ \mathbf{J}_b^T[\hat{\mathbf{e}}\hat{\mathbf{q}}, \mathbf{e}\hat{\mathbf{b}}] \end{pmatrix}$$

■ The gradient of the solution surface (General form)

$$\nabla \mathbf{f} = \mathbf{J}^T \mathbf{f}$$

$$\nabla \mathbf{f}_{g,b} = \mathbf{J}_{g,b}[\hat{\mathbf{e}}\hat{\mathbf{q}}, \mathbf{e}\hat{\mathbf{b}}]^T \mathbf{f}_{g,b}[\hat{\mathbf{e}}\hat{\mathbf{q}}, \mathbf{s}\hat{\mathbf{a}}, \mathbf{e}\hat{\mathbf{b}}, \mathbf{s}\hat{\mathbf{m}}]$$

Magnetic distortion compensation

■ Declination errors

The horizontal plane : the earth ' s surface (Heading)

■ Inclination errors

The vertical plane : the earth ' s surface (Sensor ' s attitude)

- The measured direction of the earth's magnetic field in the earth frame at time t

$${}^E\hat{\mathbf{h}}_t = (0 \quad h_x \quad h_y \quad h_z) = {}^S\hat{\mathbf{q}}_{\text{est}, t-1} \otimes {}^S\hat{\mathbf{m}}_t \otimes {}^S\hat{\mathbf{q}}_{\text{est}, t-1}^*$$

$${}^E\hat{\mathbf{b}}_t = (0 \quad \sqrt{h_x^2 + h_y^2} \quad 0 \quad h_z)$$

Gyroscope bias drift compensation

- With temperature and with motion

Karman - based approaches → estimate the gyroscope bias

Mahony et al → gyroscope bias drift (the integral feedback of the error)

The normalized direction of the estimated error in the rate of change of orientation ${}^S\dot{\mathbf{q}}_e$

- The gyroscope bias ${}^S\omega_b$

$${}^S\omega_{b,t} = \zeta \sum_t {}^S\omega_{e,t} \Delta t$$

The integral gain ζ

- DC component of ${}^S\omega_e$

$${}^S\omega_{e,t} = 2 {}^S\hat{\mathbf{q}}_{\text{est}, t-1}^* \otimes {}^S\dot{\mathbf{q}}_{e,t}$$

■ Gyroscope measurements s_{ω_c}

$$s_{\omega_c, t} = s_{\omega_t} - s_{\omega_b, t}$$

Filter gains

The estimated mean zero gyroscope measurement error of each axis $\tilde{\omega}_\beta$

■ The filter gain

$$\beta = \left| \frac{1}{2} \mathbf{q} \otimes \begin{pmatrix} 0 & \tilde{\omega}_\beta & \tilde{\omega}_\beta & \tilde{\omega}_\beta \end{pmatrix} \right| = \sqrt{\frac{3}{4}} \tilde{\omega}_\beta$$

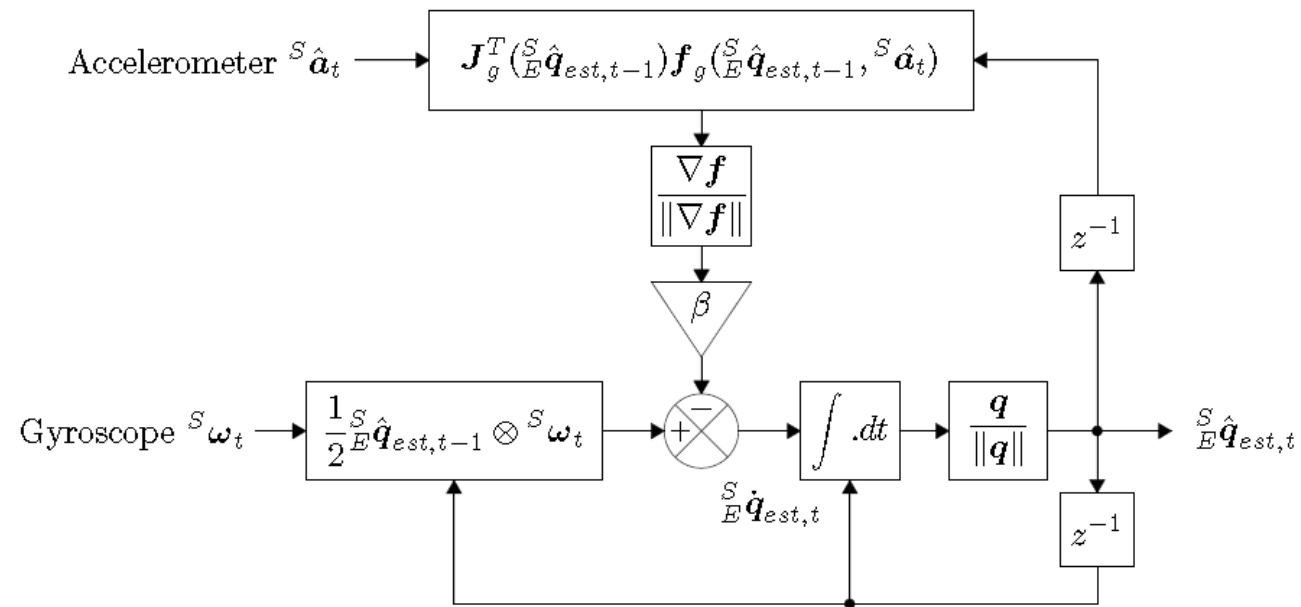
The estimated rate of gyroscope bias drift in each axis $\dot{\omega}_\xi$

■ The filter gain

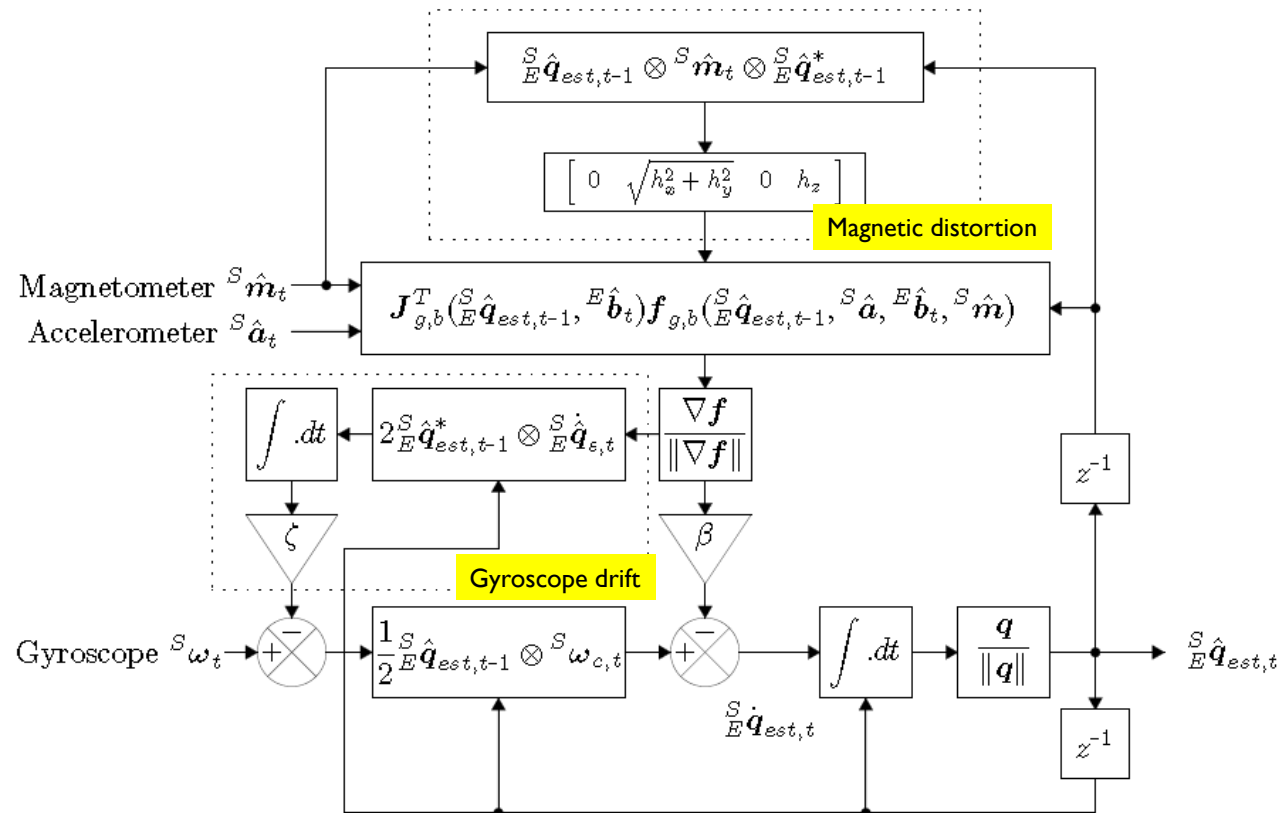
$$\xi = \xi \sqrt{\frac{3}{4}} \tilde{\omega}_\xi$$

Block diagram

■ IMU



■ MARG



MATLAB

■ Quaternion Conjugate

qConj = quaternConj (q)

qConj = (q₁ -q₂ -q₃ -q₄)

qConj=[q(:,1)-q(:,2)-q(:,3)-q(:,4)];

■ Quaternion Product

ab = quaternProd (a, b)

```
ab(:,1)=a(:,1).*b(:,1)-a(:,2).*b(:,2)-a(:,3).*b(:,3)-a(:,4).*b(:,4);
ab(:,2)=a(:,1).*b(:,2)+a(:,2).*b(:,1)+a(:,3).*b(:,4)-a(:,4).*b(:,3);
ab(:,3)=a(:,1).*b(:,3)-a(:,2).*b(:,4)+a(:,3).*b(:,1)+a(:,4).*b(:,2);
ab(:,4)=a(:,1).*b(:,4)+a(:,2).*b(:,3)-a(:,3).*b(:,2)+a(:,4).*b(:,1);
```

■ Quaternion to Rotation Matrix

R = quatern2rotMat (q)

```

R(1,1,:)= 2.*q(:,1).^2-1+2.*q(:,2).^2;
R(1,2,:)= 2.*(q(:,2).*q(:,3)+q(:,1).*q(:,4));
R(1,3,:)= 2.*(q(:,2).*q(:,4)-q(:,1).*q(:,3));
R(2,1,:)= 2.*(q(:,2).*q(:,3)-q(:,1).*q(:,4));
R(2,2,:)= 2.*q(:,1).^2-1+2.*q(:,3).^2;
R(2,3,:)= 2.*(q(:,3).*q(:,4)+q(:,1).*q(:,2));
R(3,1,:)= 2.*(q(:,2).*q(:,4)+q(:,1).*q(:,3));
R(3,2,:)= 2.*(q(:,3).*q(:,4)-q(:,1).*q(:,2));
R(3,3,:)= 2.*q(:,1).^2-1+2.*q(:,4).^2;

```

■ Quaternions to Euler

euler = quatern2euler (q)

```

R(1,1,:)=2.*q(:,1).^2-1+2.*q(:,2).^2;
R(2,1,:)=2.*(q(:,2).*q(:,3)-q(:,1).*q(:,4));
R(3,1,:)=2.*(q(:,2).*q(:,4)+q(:,1).*q(:,3));
R(3,2,:)=2.*(q(:,3).*q(:,4)-q(:,1).*q(:,2));
R(3,3,:)=2.*q(:,1).^2-1+2.*q(:,4).^2;

```

```

phi=atan2(R(3,2,:),R(3,3,:));
theta=-atan(R(3,1,:)/sqrt(1-R(3,1,:).^2));
psi=atan2(R(2,1,:),R(1,1,:));

```

```
euler=[phi(1,:) theta(1,:) psi(1,:)];
```

■ Euler to Rotation Matrix

R = euler2rotMat (phi, theta, psi)

```

R(1,1,:)=cos(psi).*cos(theta);
R(1,2,:)=sin(psi).*cos(phi)+cos(psi).*sin(theta).*sin(phi);
R(1,3,:)=sin(psi).*sin(phi)+cos(psi).*sin(theta).*cos(phi);

R(2,1,:)=sin(psi).*cos(theta);
R(2,2,:)=cos(psi).*cos(phi)+sin(psi).*sin(theta).*sin(phi);
R(2,3,:)=cos(psi).*sin(phi)+sin(psi).*sin(theta).*cos(phi);

```

```
R(3,1,:)=sin(theta);
R(3,2,:)=cos(theta).*sin(phi);
R(3,3,:)=cos(theta).*cos(phi);
```

■ Rotation Matrix to Quaternion

q = rotMat2quatern (R)

```
[row col numR]=size(R);
q=zeros(numR,4);
K=zeros(4,4);
for i=1:numR
K(1,1)=(1/3)*(R(1,1,i)-R(2,2,i)-R(3,3,i));
K(1,2)=(1/3)*(R(2,1,i)+R(1,2,i));
K(1,3)=(1/3)*(R(3,1,i)+R(1,3,i));
K(1,4)=(1/3)*(R(2,3,i)-R(3,2,i));
K(2,1)=(1/3)*(R(2,1,i)+R(1,2,i));
K(2,2)=(1/3)*(R(2,2,i)-R(1,1,i)-R(3,3,i));
K(2,3)=(1/3)*(R(3,2,i)+R(2,3,i));
K(2,4)=(1/3)*(R(3,1,i)-R(1,3,i));
K(3,1)=(1/3)*(R(3,1,i)+R(1,3,i));
K(3,2)=(1/3)*(R(3,2,i)+R(2,3,i));
K(3,3)=(1/3)*(R(3,3,i)-R(1,1,i)-R(2,2,i));
K(3,4)=(1/3)*(R(1,2,i)-R(2,1,i));
K(4,1)=(1/3)*(R(2,3,i)-R(3,2,i));
K(4,2)=(1/3)*(R(3,1,i)-R(1,3,i));
K(4,3)=(1/3)*(R(1,2,i)-R(2,1,i));
K(4,4)=(1/3)*(R(1,1,i)+R(2,2,i)+R(3,3,i));
[V,D]=eig(K);
%p=find(max(D));
%q=V(:,p);
q(i,:)=V(:,4);
q(i,:)=[q(i,4) q(i,1) q(i,2) q(i,3)];
end
```

■ Axis Angle to Quaternion

q = axisAngle2quatern (axis, angle)

```
q0=cos(angle./2);  
q1=-axis(:,1)*sin(angle./2);  
q2=-axis(:,2)*sin(angle./2);  
q3=-axis(:,3)*sin(angle./2);  
q=[q0 q1 q2 q3];
```

■ Rotation Matrix to Euler

euler = rotMat2euler (R)

```
phi=atan2(R(3,2,:),R(3,3,:));  
theta=-atan(R(3,1,:)/sqrt(1-R(3,1,:).^2));  
psi=atan2(R(2,1,:),R(1,1,:));  
  
euler=[phi(1,:) theta(1,:) psi(1,:)];
```

■ Axis Angle to Rotation Matrix

R = axisAngle2rotMat (axis, angle)

```
kx=axis(:,1);  
ky=axis(:,2);  
kz=axis(:,3);  
cT=cos(angle);  
sT=sin(angle);  
vT=1-cos(angle);  
  
R(1,1,:)=kx.*kx.*vT+cT;  
R(1,2,:)=kx.*ky.*vT-kz.*sT;  
R(1,3,:)=kx.*kz.*vT+ky.*sT;
```

```
R(2,1,:)=kx.*ky.*vT+kz.*sT;
R(2,2,:)=ky.*ky.*vT+cT;
R(2,3,:)=ky.*kz.*vT-kx.*sT;
```

```
R(3,1,:)=kx.*kz.*vT-ky.*sT;
R(3,2,:)=ky.*kz.*vT+kx.*sT;
R(3,3,:)=kz.*kz.*vT+cT;
```

■ Update

```
obj = Update (obj, Gyroscope, Accelerometer, Magnetometer)
```

■ % short name local variable for readability

```
q = obj.Quaternion;
```

■ % Normalise accelerometer measurement

```
if (norm (Accelerometer) == 0), return; end % handle NaN
Accelerometer = Accelerometer / norm (Accelerometer); % normalise magnitude
```

■ % Normalise magnetometer measurement

```
if (norm (Magnetometer) == 0), return; end % handle NaN
Magnetometer = Magnetometer / norm (Magnetometer); % normalise magnitude
```

■ % Reference direction of Earth's magnetic feild

```
h = quaternProd (q, quaternProd ([0 Magnetometer], quaternConj (q)));
b = [0 norm ([h (2) h (3)]) 0 h (4)];
```

■ % Gradient decent algorithm corrective step

$$F = \begin{pmatrix} 2 * (q(2) * q(4) - q(1) * q(3)) - \text{Accelerometer}(1) \\ 2 * (q(1) * q(2) + q(3) * q(4)) - \text{Accelerometer}(2) \\ 2 * (0.5 - q(2)^2 - q(3)^2) - \text{Accelerometer}(3) \\ 2 * b(2) * (0.5 - q(3)^2 - q(4)^2) + 2 * b(4) * (q(2) * q(4) - q(1) * q(3)) - \text{Magnetometer}(1) \\ 2 * b(2) * (q(2) * q(3) - q(1) * q(4)) + 2 * b(4) * (q(1) * q(2) + q(3) * q(4)) - \text{Magnetometer}(2) \\ 2 * b(2) * (q(1) * q(3) + q(2) * q(4)) + 2 * b(4) * (0.5 - q(2)^2 - q(3)^2) - \text{Magnetometer}(3) \end{pmatrix}$$

$$J = \begin{pmatrix} -2 * q(3) & 2 * q(4) & -2 * q(1) & 2 * q(2) \\ 2 * q(2) & 2 * q(1) & 2 * q(4) & 2 * q(3) \\ 0 & -4 * q(2) & -4 * q(3) & 0 \\ -2 * b(4) * q(3) & 2 * b(4) * q(4) & -4 * b(2) * q(3) - 2 * b(4) * q(1) & -4 * b(2) * q(4) + 2 * b(4) * q(2) \\ -2 * b(2) * q(4) + 2 * b(4) * q(2) & 2 * b(2) * q(3) + 2 * b(4) * q(1) & 2 * b(2) * q(2) + 2 * b(4) * q(4) & -2 * b(2) * q(1) + 2 * b(4) * q(3) \\ 2 * b(2) * q(3) & 2 * b(2) * q(4) - 4 * b(4) * q(2) & 2 * b(2) * q(1) - 4 * b(4) * q(3) & 2 * b(2) * q(2) \end{pmatrix}$$

step = (J' * F);

■ % normalise step magnitude

step = step / norm (step);

■ % Compute rate of change of quaternion

qDot = 0.5 * quaternProd (q, [0 Gyroscope (1) Gyroscope (2) Gyroscope (3)]) - obj.Beta * step';

■ % Integrate to yield quaternion

q = q + qDot * obj.SamplePeriod;

■ % normalise quaternion

obj.Quaternion = q / norm (q);

■ IMU

```
function obj=UpdateIMU(obj,Gyroscope,Accelerometer)
q=obj.Quaternion;% short name local variable for readability

% Normalise accelerometer measurement
if(norm(Accelerometer)==0),return;end % handle NaN
Accelerometer=Accelerometer/norm(Accelerometer);% normalise magnitude

% Gradient decent algorithm corrective step
F=[2*(q(2)*q(4)-q(1)*q(3))-Accelerometer(1) 2*(q(1)*q(2)+q(3)*q(4))-Accelerometer(2) 2*(0.5-q(2)^2-q(3)^2)-Accelerometer(3)];
J=[-2*q(3),2*q(4),-2*q(1),2*q(2) 2*q(2),2*q(1),2*q(4),2*q(3) 0,-4*q(2),-4*q(3),0];
step=(J*F);
step=step/norm(step);% normalise step magnitude

% Compute rate of change of quaternion
qDot=0.5*quaternProd(q,[0 Gyroscope(1) Gyroscope(2) Gyroscope(3)])-obj.Beta*step';

% Integrate to yield quaternion
q=q+qDot*obj.SamplePeriod;
obj.Quaternion=q/norm(q);% normalise quaternion
end
```

Kalman Filter

■ Gyro Angler Rate (p, q, r) → Euler Angle 변화율 ($\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$)

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin[\phi] \tan[\theta] & \cos[\phi] \tan[\theta] \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] / \cos[\theta] & \cos[\phi] / \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} \phi + dt * \dot{\phi} \\ \theta + dt * \dot{\theta} \\ \varphi + dt * \dot{\varphi} \end{pmatrix}$$

■ Accelerometer (f_x, f_y, f_z)

Linear Rate (u, v, w), Linear Acceleration ($\dot{u}, \dot{v}, \dot{w}$), Angler Rate (p, q, r), Gravity (g)

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} + \begin{pmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} + g \begin{pmatrix} \sin[\theta] \\ -\cos[\theta] \sin[\phi] \\ -\cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\therefore \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = g \begin{pmatrix} \sin[\theta] \\ -\cos[\theta] \sin[\phi] \\ -\cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\theta = \text{ArcSin}\left[\frac{f_x}{g}\right]$$

$$\phi = \text{ArcSin}\left[-\frac{f_y}{g \cos[\theta]}\right]$$

■ Sensor Fusion

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{pmatrix} \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}_{k+1} = \left(\mathbf{I} + \Delta t \cdot \frac{1}{2} \begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{pmatrix} \right) \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}_k$$

$$\mathbf{A} = \mathbf{I} + \Delta t \cdot \frac{1}{2} \begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] + \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ \sin\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] - \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] + \sin\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ \cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] - \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] \end{pmatrix}$$

■ Euler Angle

ϕ around $\hat{x}_B$, θ around $\hat{y}_B$, ψ around $\hat{z}_B$

$$\phi = \text{ArcTan}[1 - 2 (q_2^2 + q_3^2), 2 (q_3 q_4 + q_1 q_2)]$$

$$\theta = -\text{ArcSin}[2 (q_2 q_4 - q_1 q_3)]$$

$$\psi = \text{ArcTan}[1 - 2 (q_3^2 + q_4^2), 2 (q_2 q_3 + q_1 q_4)]$$

■ Algorithm start

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Q} = \begin{pmatrix} 0.0001 & 0 & 0 & 0 \\ 0 & 0.0001 & 0 & 0 \\ 0 & 0 & 0.0001 & 0 \\ 0 & 0 & 0 & 0.0001 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

■ KF

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \mathbf{x}$$

$$\mathbf{z} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] + \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ \sin\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] - \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ \cos\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] + \sin\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] \\ \cos\left[\frac{\phi}{2}\right] \cos\left[\frac{\theta}{2}\right] \sin\left[\frac{\psi}{2}\right] - \sin\left[\frac{\phi}{2}\right] \sin\left[\frac{\theta}{2}\right] \cos\left[\frac{\psi}{2}\right] \end{pmatrix}$$

$$\mathbf{x}_p = \mathbf{A} \cdot \mathbf{x}$$

$$\mathbf{A} = \mathbf{I} + \Delta t \cdot \frac{1}{2} \begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{pmatrix}$$

$$\mathbf{P}_p = \mathbf{A} \cdot \mathbf{P} \cdot \mathbf{A}^T + \mathbf{Q}$$

$$\mathbf{K} = \mathbf{P}_p \cdot \mathbf{H}^T \cdot \text{Inverse}[\mathbf{H} \cdot \mathbf{P}_p \cdot \mathbf{H}^T + \mathbf{R}]$$

$$\mathbf{x} = \mathbf{x}_p + \mathbf{K} \cdot (\mathbf{z} - \mathbf{H} \cdot \mathbf{x}_p)$$

$$\mathbf{P} = \mathbf{P}_p - \mathbf{K} \cdot \mathbf{H} \cdot \mathbf{P}_p$$

$$\begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} \text{ArcTan}[1 - 2 (q_2^2 + q_3^2), 2 (q_3 q_4 + q_1 q_2)] \\ -\text{ArcSin}[2 (q_2 q_4 - q_1 q_3)] \\ \text{ArcTan}[1 - 2 (q_3^2 + q_4^2), 2 (q_2 q_3 + q_1 q_4)] \end{pmatrix}$$

Essential Kalman Filter

$$\mathbf{x} = \begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix}$$

$$\mathbf{rates} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{r} \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\varphi} \end{pmatrix} &= \begin{pmatrix} 1 & \sin[\phi] \tan[\theta] & \cos[\phi] \tan[\theta] \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] / \cos[\theta] & \cos[\phi] / \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{r} \end{pmatrix} + \mathbf{w} \\ &= \begin{pmatrix} \mathbf{p} + \mathbf{r} \cos[\phi] \tan[\theta] + \mathbf{q} \sin[\phi] \tan[\theta] \\ \mathbf{q} \cos[\phi] - \mathbf{r} \sin[\phi] \\ \mathbf{r} \cos[\phi] \sec[\theta] + \mathbf{q} \sec[\theta] \sin[\phi] \end{pmatrix} + \mathbf{w} \\ &= \mathbf{f}[\mathbf{x}] + \mathbf{w} \end{aligned}$$

$$\begin{aligned} \mathbf{z} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix} + \mathbf{v} \\ &= \mathbf{H} \mathbf{x} + \mathbf{v} \end{aligned}$$

- **Algorithm start**

$$\mathbf{A} = \begin{pmatrix} \frac{\partial f_1}{\partial \phi} & \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \varphi} \\ \frac{\partial f_2}{\partial \phi} & \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \varphi} \\ \frac{\partial f_3}{\partial \phi} & \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial \varphi} \end{pmatrix}$$

$$\mathbf{H} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}$$

$$Q = \begin{pmatrix} 0.0001 & 0 & 0 \\ 0 & 0.0001 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

- EKF

$$\begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix} = \mathbf{x}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \text{rates}$$

```
A = Ajacob[x, rates, dt]
```

$$A = \begin{pmatrix} q \cos[\phi] \tan[\theta] - r \sin[\phi] \tan[\theta] & q \sin[\phi] \sec[\theta]^2 + r \cos[\phi] \sec[\theta]^2 & 0 \\ -q \sin[\phi] - r \cos[\phi] & 0 & 0 \\ q \cos[\phi] \sec[\theta] - r \sin[\phi] \sec[\theta] & q \sin[\phi] \sec[\theta] \tan[\theta] + r \cos[\phi] \sec[\theta] \tan[\theta] & 0 \end{pmatrix}$$

$$A = I + \Delta t * A$$

$$xp = fx[x, rates, dt]$$

$$xp = fx[x, rates, dt] = \begin{pmatrix} p + r \cos[\phi] \tan[\theta] + q \sin[\phi] \tan[\theta] \\ q \cos[\phi] - r \sin[\phi] \\ r \cos[\phi] \sec[\theta] + q \sec[\theta] \sin[\phi] \end{pmatrix}$$

$$Pp = A.P.A^T + Q$$

$$K = Pp.H^T.Inverse[H.Pp.H^T + R]$$

$$x = xp + K.(z - H * xp)$$

$$P = Pp - K.H.Pp$$