

Rotation Matrix

$$R_x[\phi] := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$R_y[\theta] := \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} // \text{MatrixForm}$$

$$R_z[\psi] := \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} // \text{MatrixForm}$$

■ Euler Angle

$$R = R_z[\psi] \cdot R_y[\theta] \cdot R_x[\phi] = \begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\phi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$R_z[\psi] \cdot R_y[\theta] \cdot R_x[\phi]$$

$$\begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] - \cos[\phi] \sin[\psi] & \cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ \cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] + \sin[\theta] \sin[\phi] \sin[\psi] & -\cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\begin{pmatrix} \cos[\psi] & \sin[\psi] & 0 \\ -\sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & -\sin[\theta] \\ 0 & 1 & 0 \\ \sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & \sin[\phi] \\ 0 & -\sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] + \cos[\phi] \sin[\psi] & -\cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ -\cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi] & \cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ \sin[\theta] & -\cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

Rotation matrix from axis and angle

$$R[\theta] :=$$

$$\begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_y u_x (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_z u_x (1 - \cos[\theta]) - u_y \sin[\theta] & u_z u_y (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix}$$

■ concisely

$$R = I \cdot \cos[\theta] + \sin[\theta] u_x + (1 - \cos[\theta]) u \otimes u$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{u}_x = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}$$

$$\mathbf{u} \otimes \mathbf{u} = \begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{pmatrix}$$

■ ex

$\{u_x, u_y, u_z\} = \{0, 0, 1\};$

$$\begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_y u_x (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_z u_x (1 - \cos[\theta]) - u_y \sin[\theta] & u_z u_y (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix}.$$

$$\begin{pmatrix} Ax \\ Ay \\ Az \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} Ax \cos[\theta] - Ay \sin[\theta] \\ Ay \cos[\theta] + Ax \sin[\theta] \\ Az \end{pmatrix}$$

$$\begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_y u_x (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_z u_x (1 - \cos[\theta]) - u_y \sin[\theta] & u_z u_y (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix}.$$

$$\begin{pmatrix} Ax \\ Ay \\ Az \end{pmatrix} // \text{FullSimplify} // \text{InputForm}$$

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{Ax*Cos[θ] + Az*Sin[θ]*Subscript[u, y] - (-1 + Cos[θ])*Subscript[u, x]*
(Ax*Subscript[u, x] + Ay*Subscript[u, y]) -
(Ay*Sin[θ] + Az*(-1 + Cos[θ])*Subscript[u, x])*Subscript[u, z]},
{Ay*Cos[θ] - Subscript[u, x]*(Az*Sin[θ] + Ax*(-1 + Cos[θ])*Subscript[u, y]) +
Ax*Sin[θ]*Subscript[u, z] - (-1 + Cos[θ])*Subscript[u, y]*
(Ay*Subscript[u, y] + Az*Subscript[u, z])},
{Az*Cos[θ] + Ay*Sin[θ]*Subscript[u, x] - Ax*Sin[θ]*Subscript[u, y] -
(-1 + Cos[θ])*(Ax*Subscript[u, x] + Ay*Subscript[u, y])*Subscript[u, z] -
Az*(-1 + Cos[θ])*Subscript[u, z]^2}
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$R_x[\phi_] := // \text{MatrixForm}$

$$R_y[\theta_] := \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} // \text{MatrixForm}$$

$$R_z[\psi_] := \begin{pmatrix} \cos[\psi] & -\sin[\psi] & 0 \\ \sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[y] & -\sin[y] & 0 \\ \sin[y] & \cos[y] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[p] & 0 & \sin[p] \\ 0 & 1 & 0 \\ -\sin[p] & 0 & \cos[p] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[r] & -\sin[r] \\ 0 & \sin[r] & \cos[r] \end{pmatrix} \cdot \begin{pmatrix} Wx \\ Wy \\ Wz \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} Wx \cos[p] \cos[y] + Wy (\cos[y] \sin[p] \sin[r] - \cos[r] \sin[y]) + Wz (\cos[r] \cos[y] \sin[p] + \sin[r] \sin[y]) \\ Wx \cos[p] \sin[y] + Wz (-\cos[y] \sin[r] + \cos[r] \sin[p] \sin[y]) + Wy (\cos[r] \cos[y] + \sin[p] \sin[r]) \\ Wz \cos[p] \cos[r] - Wx \sin[p] + Wy \cos[p] \sin[r] \end{pmatrix}$$

$$\begin{pmatrix} \cos[p] & 0 & \sin[p] \\ 0 & 1 & 0 \\ -\sin[p] & 0 & \cos[p] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[r] & -\sin[r] \\ 0 & \sin[r] & \cos[r] \end{pmatrix} \cdot \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$\{ \{V_x \cos[p] + V_z \cos[r] \sin[p] + V_y \sin[p] \sin[r]\}, \\ \{V_y \cos[r] - V_z \sin[r]\}, \{V_z \cos[p] \cos[r] - V_x \sin[p] + V_y \cos[p] \sin[r]\} \}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[r] & -\sin[r] \\ 0 & \sin[r] & \cos[r] \end{pmatrix} \cdot \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix}$$

$$\{ \{U_x\}, \{U_y \cos[r] - U_z \sin[r]\}, \{U_z \cos[r] + U_y \sin[r]\} \}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[-r] & -\sin[-r] \\ 0 & \sin[-r] & \cos[-r] \end{pmatrix} \cdot \begin{pmatrix} \cos[-p] & 0 & \sin[-p] \\ 0 & 1 & 0 \\ -\sin[-p] & 0 & \cos[-p] \end{pmatrix} \cdot \begin{pmatrix} \cos[-y] & -\sin[-y] & 0 \\ \sin[-y] & \cos[-y] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X_x \\ X_y \\ X_z \end{pmatrix}$$

$$\{ \{X_x \cos[p] \cos[y] - X_z \sin[p] + X_y \cos[p] \sin[y]\}, \\ \{X_z \cos[p] \sin[r] + X_x (\cos[y] \sin[p] \sin[r] - \cos[r] \sin[y]) + \\ X_y (\cos[r] \cos[y] + \sin[p] \sin[r] \sin[y])\}, \{X_z \cos[p] \cos[r] + \\ X_y (-\cos[y] \sin[r] + \cos[r] \sin[p] \sin[y]) + X_x (\cos[r] \cos[y] \sin[p] + \sin[r] \sin[y])\} \}$$