

Roll: ϕ **Pitch: θ**

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} + g \begin{pmatrix} \sin[\theta] \\ -\cos[\theta] \sin[\phi] \\ -\cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} \cos[\theta] & \sin[\theta] \sin[\phi] & \cos[\phi] \sin[\theta] \\ 0 & \cos[\phi] & -\sin[\phi] \\ -\sin[\theta] & \cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & -g \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & \sin[\theta] \\ 0 & 1 & 0 \\ -\sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & -\sin[\phi] \\ 0 & \sin[\phi] & \cos[\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} g \sin[\theta] & -g \cos[\theta] \sin[\phi] & -g \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[-\phi] & -\sin[-\phi] \\ 0 & \sin[-\phi] & \cos[-\phi] \end{pmatrix} \cdot \begin{pmatrix} \cos[-\theta] & 0 & \sin[-\theta] \\ 0 & 1 & 0 \\ -\sin[-\theta] & 0 & \cos[-\theta] \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} g \sin[\theta] \\ -g \cos[\theta] \sin[\phi] \\ -g \cos[\theta] \cos[\phi] \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[-\phi] & -\sin[-\phi] \\ 0 & \sin[-\phi] & \cos[-\phi] \end{pmatrix} // \text{MatrixForm}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & \sin[\phi] \\ 0 & -\sin[\phi] & \cos[\phi] \end{pmatrix}$$

yaw: φ

$$\mathbf{x} = \begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix}$$

$$\begin{pmatrix} \phi \\ \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} 1 & \sin[\phi] \tan[\theta] & \cos[\phi] \tan[\theta] \\ 0 & \cos[\phi] & -\sin[\theta] \\ 0 & \sin[\phi] \sec[\theta] & \cos[\phi] \sec[\theta] \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\tan[\theta] = \frac{\sin[\theta]}{\cos[\theta]}$$

$$\text{Sec}[\theta] = \frac{1}{\cos[\theta]}$$

Euler Angle

Rotation Matrix

■ 3D orthonormal basis → unit vectors

$$\hat{u}, \hat{v}, \hat{w}$$

$$A = \begin{pmatrix} \hat{u}_x & \hat{v}_x & \hat{w}_x \\ \hat{u}_y & \hat{v}_y & \hat{w}_y \\ \hat{u}_z & \hat{v}_z & \hat{w}_z \end{pmatrix}$$

$$|\hat{u}| = |\hat{v}| = 1$$

$$\hat{u} \cdot \hat{v} = 0$$

$$\hat{u} \times \hat{v} = \hat{w}$$

■ single rotation about some axis (unit vector)

■ rotation vector

$$\hat{e} = \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} \Rightarrow \mathbf{v} = \theta \hat{e}$$

■ Quaternions

■ normalized four-dimensional vector

$$\hat{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}$$

■ Euler axis

$$\hat{e} = \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}$$

■ Angle

$$\theta$$

■ vector's elements

$$q_1 = e_x \sin\left[\frac{\theta}{2}\right]$$

$$q_2 = e_y \sin\left[\frac{\theta}{2}\right]$$

$$q_3 = e_z \sin\left[\frac{\theta}{2}\right]$$

$$q_4 = \cos\left[\frac{\theta}{2}\right]$$

■ the quaternion parametrization obeys

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$$

■ hypercomplex numbers {i,j,k}

$$a + b i + c j + d k$$

$$i^2 = j^2 = k^2 = -1$$

$$i j = -j i = k$$

$$j k = -k j = i$$

$$k i = -i k = j$$

■ Quaternion multiplication

$$\tilde{q} \otimes q = \begin{pmatrix} q_4 & -q_3 & -q_2 & -q_1 \\ -q_3 & q_4 & -q_1 & q_2 \\ -q_2 & -q_1 & q_4 & q_3 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix} \cdot \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \\ \tilde{q}_4 \end{pmatrix}$$

$$A_3 = A_2 A_1$$

$$q_3 = q_1 \otimes q_2$$

Rotation matrix ↔ Euler angles

3 - 1 - 3 Euler angles ϕ, θ, ψ (around Z, X, Z axis)

Euler angles (ϕ, θ, ψ)

$$A = \begin{pmatrix} \hat{u}_x & \hat{v}_x & \hat{w}_x \\ \hat{u}_y & \hat{v}_y & \hat{w}_y \\ \hat{u}_z & \hat{v}_z & \hat{w}_z \end{pmatrix}$$

$$\phi = \text{ArcTan}\left[\frac{A_{31}}{A_{32}}\right]$$

$$\theta = \text{ArcCos}[A_{33}]$$

$$\psi = -\text{ArcTan}\left[\frac{A_{13}}{A_{23}}\right]$$

$$\mathbf{A} = \mathbf{A}_3 \mathbf{A}_2 \mathbf{A}_1 = \mathbf{A}_Z \mathbf{A}_Y \mathbf{A}_X$$

$$\mathbf{A}_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & \sin[\phi] \\ 0 & -\sin[\phi] & \cos[\phi] \end{pmatrix}$$

$$\mathbf{A}_Y = \begin{pmatrix} \cos[\theta] & 0 & -\sin[\theta] \\ 0 & 1 & 0 \\ \sin[\theta] & 0 & \cos[\theta] \end{pmatrix}$$

$$\mathbf{A}_Z = \begin{pmatrix} \cos[\psi] & \sin[\psi] & 0 \\ -\sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \cos[\psi] & \sin[\psi] & 0 \\ -\sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta] & 0 & -\sin[\theta] \\ 0 & 1 & 0 \\ \sin[\theta] & 0 & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & \sin[\phi] \\ 0 & -\sin[\phi] & \cos[\phi] \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \cos[\theta] \cos[\psi] & \cos[\psi] \sin[\theta] \sin[\phi] + \cos[\phi] \sin[\psi] & -\cos[\phi] \cos[\psi] \sin[\theta] + \sin[\phi] \sin[\psi] \\ -\cos[\theta] \sin[\psi] & \cos[\phi] \cos[\psi] - \sin[\theta] \sin[\phi] \sin[\psi] & \cos[\psi] \sin[\phi] + \cos[\phi] \sin[\theta] \sin[\psi] \\ \sin[\theta] & -\cos[\theta] \sin[\phi] & \cos[\theta] \cos[\phi] \end{pmatrix}$$

Rotation matrix

■ Decomposing

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{32} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} \cos[\theta_y] \cos[\theta_z] & \cos[\theta_z] \sin[\theta_x] \sin[\theta_y] - \cos[\theta_x] \sin[\theta_z] & \cos[\theta_x] \cos[\theta_z] \sin[\theta_y] \\ \cos[\theta_y] \sin[\theta_z] & \cos[\theta_x] \cos[\theta_z] + \sin[\theta_x] \sin[\theta_y] \sin[\theta_z] & -\cos[\theta_z] \sin[\theta_x] + \cos[\theta_x] \cos[\theta_y] \\ -\sin[\theta_y] & \cos[\theta_y] \sin[\theta_x] & \cos[\theta_x] \cos[\theta_y] \end{pmatrix}$$

$$\theta_x = \text{ArcTan}\left[\frac{r_{32}}{r_{33}}\right]$$

$$\theta_y = \text{ArcTan}\left[-\frac{r_{31}}{\sqrt{r_{32}^2 + r_{33}^2}}\right]$$

$$\theta_z = \text{ArcTan}\left[\frac{r_{21}}{r_{11}}\right]$$

■ Composing

$$\mathbf{X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\theta_x] & -\sin[\theta_x] \\ 0 & \sin[\theta_x] & \cos[\theta_x] \end{pmatrix}$$

$$\mathbf{Y} = \begin{pmatrix} \cos[\theta_y] & 0 & \sin[\theta_y] \\ 0 & 1 & 0 \\ -\sin[\theta_y] & 0 & \cos[\theta_y] \end{pmatrix}$$

$$Z = \begin{pmatrix} \cos[\theta_z] & -\sin[\theta_z] & 0 \\ \sin[\theta_z] & \cos[\theta_z] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R = ZYX$$

$$R = \begin{pmatrix} \cos[\theta_z] & -\sin[\theta_z] & 0 \\ \sin[\theta_z] & \cos[\theta_z] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\theta_y] & 0 & \sin[\theta_y] \\ 0 & 1 & 0 \\ -\sin[\theta_y] & 0 & \cos[\theta_y] \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\theta_x] & -\sin[\theta_x] \\ 0 & \sin[\theta_x] & \cos[\theta_x] \end{pmatrix}$$

$$\theta_x \rightarrow \{-\pi, \pi\}$$

$$\theta_y \rightarrow \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$$

$$\theta_z \rightarrow \{-\pi, \pi\}$$

2D Rotation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos[\theta] & -\sin[\theta] \\ \sin[\theta] & \cos[\theta] \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \cos[\theta] - y \sin[\theta] \\ y \cos[\theta] + x \sin[\theta] \end{pmatrix}$$

Rotation matrix ↔ Euler axis/angles

$$\hat{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$$\theta = \text{ArcCos}\left[\frac{1}{2} (A_{11} + A_{22} + A_{33} - 1)\right]$$

$$e_1 = \frac{A_{32} - A_{23}}{2 \sin[\theta]}$$

$$e_2 = \frac{A_{13} - A_{31}}{2 \sin[\theta]}$$

$$e_3 = \frac{A_{21} - A_{12}}{2 \sin[\theta]}$$

General rotations

$$R_X[\gamma] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\gamma] & -\sin[\gamma] \\ 0 & \sin[\gamma] & \cos[\gamma] \end{pmatrix} : \text{roll } \gamma$$

$$R_Y[\beta] = \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} : \text{pitch } \beta$$

$$R_Z[\alpha] = \begin{pmatrix} \cos[\alpha] & -\sin[\alpha] & 0 \\ \sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} : \text{yaw } \alpha$$

$$R = R_X[\gamma] R_Y[\beta] R_Z[\alpha]$$

Euler Angle

$$\mathbf{R} = \mathbf{R}_z[\gamma] \mathbf{R}_x[\beta] \mathbf{R}_y[\alpha]$$

Rotation matrix from axis and angle

■ Euler axis

$$\hat{\mathbf{e}} = \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix}$$

■ Angle

$$\theta$$

■ Rotation

$$\mathbf{R} = \mathbf{I} \cos[\theta] + \sin[\theta] [\mathbf{u}]_{\times} + (1 - \cos[\theta]) \mathbf{u} \otimes \mathbf{u}$$

$$\mathbf{u} \otimes \mathbf{u} = \begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{pmatrix}$$

$$[\mathbf{u}]_{\times} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}$$

$$\mathbf{R} =$$

$$\begin{pmatrix} \cos[\theta] + u_x^2 (1 - \cos[\theta]) & u_x u_y (1 - \cos[\theta]) - u_z \sin[\theta] & u_x u_z (1 - \cos[\theta]) + u_y \sin[\theta] \\ u_x u_y (1 - \cos[\theta]) + u_z \sin[\theta] & \cos[\theta] + u_y^2 (1 - \cos[\theta]) & u_y u_z (1 - \cos[\theta]) - u_x \sin[\theta] \\ u_x u_z (1 - \cos[\theta]) - u_y \sin[\theta] & u_y u_z (1 - \cos[\theta]) + u_x \sin[\theta] & \cos[\theta] + u_z^2 (1 - \cos[\theta]) \end{pmatrix}$$