

Fourier Transform $\{x[\xi], \hat{x}[\xi], F\{x\}[\xi]\}$

■ Discrete Time Fourier Transform (DTFT)

$$X[f] = \int_{-\infty}^{\infty} x[t] e^{-i 2 \pi f t} dt$$

$$X[f] = \int_{-\infty}^{\infty} x[t] (\cos[2 \pi f t] - i \sin[2 \pi f t]) dt = \int_{-\infty}^{\infty} x[t] e^{-i 2 \pi f t} dt$$

t : Time, f : frequency

$$x[t] = \int_{-\infty}^{\infty} X[f] e^{i 2 \pi f t} df$$

$$\{ \therefore e^{it} = \cos[t] + i \sin[t] \}$$

■ Discrete Fourier Transform (DFT)

$$S[k] = \frac{1}{N} \sum_{n=0}^{N-1} (s[n] e^{-i 2 \pi \frac{k}{N} n}) = \frac{1}{N} \sum_{n=0}^{N-1} (s[n] X_N^{kn})$$

$$X_N = e^{-i \frac{2 \pi}{N}}$$

$$s[n] = \frac{1}{N} \sum_{k=0}^{N-1} (S[k] e^{i 2 \pi \frac{k}{N} n}) = \frac{1}{N} \sum_{k=0}^{N-1} (S[k] X_N^{-kn})$$

$$S[k + N] = S[k]$$

■ Fast Fourier Transform (FFT)

$$N = 2^n$$

$$N = 2 L$$

$$\begin{aligned} S[k] &= \frac{1}{2 L} \sum_{n=0}^{2 L-1} (s[n] e^{-i \frac{2 \pi k n}{2 L}}) = \frac{1}{2 L} \sum_{n=0}^{2 L-1} (s[n] X_{2 L}^{kn}) \\ &= \frac{1}{2} \left(\frac{1}{L} \sum_{n=0}^{L-1} (s[2 n] X_{2 L}^{k(2 n)}) + \frac{1}{L} \sum_{n=0}^{L-1} (s[2 n + 1] X_{2 L}^{k(2 n+1)}) \right) \end{aligned}$$

$$X_{2 L}^{k(2 n)} = e^{-i \frac{2 \pi k (2 n)}{2 L}} = e^{-i \frac{2 \pi k (n)}{L}} = X_L^{k(n)}$$

$$X_{2 L}^{k(2 n+1)} = X_{2 L}^{k(2 n)} X_{2 L}^k = X_L^{k(n)} X_{2 L}^k$$

$$S[k] = \frac{1}{2} \left(\frac{1}{L} \sum_{n=0}^{L-1} (s[2 n] X_L^{k(n)}) + \frac{1}{L} \sum_{n=0}^{L-1} (s[2 n + 1] X_L^{k(n)} X_{2 L}^k) \right)$$

$$S_{\text{even}}[k] = \frac{1}{L} \sum_{n=0}^{L-1} (s[2 n] X_L^{kn})$$

$$S_{\text{odd}}[k] = \frac{1}{L} \sum_{n=0}^{L-1} (s[2n+1] X_L^{kn})$$

$$S[k] = \frac{1}{2} (S_{\text{even}}[k] + S_{\text{odd}}[k] X_{2L}^k)$$

$$\begin{aligned} S[k+L] &= \frac{1}{2} \left(\frac{1}{L} \sum_{n=0}^{L-1} (s[2n] X_L^{(k+L)(n)}) + \frac{1}{L} \sum_{n=0}^{L-1} (s[2n+1] X_L^{(k+L)(n)} X_{2L}^{(k+L)}) \right) \\ &= \frac{1}{2} (S_{\text{even}}[k+L] + S_{\text{odd}}[k+L] X_{2L}^{(k+L)}) = \frac{1}{2} (S_{\text{even}}[k] - S_{\text{odd}}[k] X_{2L}^k) \end{aligned}$$

$$X_L^{k+L} = e^{-\frac{2\pi k(1+L)}{L}i} = e^{-\frac{2\pi k}{L}i} e^{-2\pi k i} = e^{-\frac{2\pi k}{L}i} = X_L^k$$

$$X_{2L}^{k+L} = e^{-\frac{2\pi k(1+L)}{2L}i} = e^{-\frac{2\pi k}{2L}i} e^{-\pi k i} = e^{-\frac{2\pi k}{2L}i} = -X_{2L}^k$$

$$f_j = \sum_{k=0}^{n-1} x_k e^{-\frac{2\pi i}{n} j k}$$

■ Discrete Cosine Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} (x[n] e^{-i 2\pi \frac{k}{N} n}) = \sum_{n=0}^{N-1} (x[n] W_N^{kn})$$

$$W_N = e^{-i \frac{2\pi}{N}}$$

$$x[n] = \sum_{k=0}^{N-1} (X[k] e^{i 2\pi \frac{k}{N} n}) = \frac{1}{N} \sum_{k=0}^{N-1} (X[k] W_N^{-kn})$$

$$X[k+N] = X[k]$$

$$X_{2L}^{k(2n)} = e^{-\frac{2\pi k(2n)}{2L}i} = e^{-\frac{2\pi k(n)}{L}i} = X_L^{k(n)}$$

$$X_{2L}^{k(2n+1)} = X_{2L}^{k(2n)} X_{2L}^k = X_L^{k(n)} X_{2L}^k$$

$$X_L^{k+L} = e^{-\frac{2\pi k(1+L)}{L}i} = e^{-\frac{2\pi k}{L}i} e^{-2\pi k i} = e^{-\frac{2\pi k}{L}i} = X_L^k$$

$$X_{2L}^{k+L} = e^{-\frac{2\pi k(1+L)}{2L}i} = e^{-\frac{2\pi k}{2L}i} e^{-\pi k i} = e^{-\frac{2\pi k}{2L}i} = -X_{2L}^k$$

■ N=2

$$X[k] = \sum_{n=0}^{N-1} (x[n] e^{-i 2\pi \frac{k}{N} n}) = \sum_{n=0}^{N-1} (x[n] W_N^{kn})$$

$$X[0] = x[0] W_2^{0*0} + x[1] W_2^{0*1} = x[0] W_2^0 + x[1] W_2^0$$

$$X[1] = x[0] W_2^{0*1} + x[1] W_2^{1*1} = x[0] W_2^0 + x[1] W_2^1$$

■ N=4

$$X[k] = \sum_{n=0}^{N-1} (x[n] e^{-i 2\pi \frac{k}{N} n}) = \sum_{n=0}^{N-1} (x[n] W_N^{kn})$$

$$X[0] = x[0] W_4^{0*0} + x[1] W_4^{0*1} + x[2] W_4^{0*2} + x[3] W_4^{0*3}$$

$$X[1] = x[0] W_4^{1*0} + x[1] W_4^{1*1} + x[2] W_4^{1*2} + x[3] W_4^{1*3}$$

$$X[2] = x[0] W_4^{2*0} + x[1] W_4^{2*1} + x[2] W_4^{2*2} + x[3] W_4^{2*3}$$

$$X[3] = x[0] W_4^{3*0} + x[1] W_4^{3*1} + x[2] W_4^{3*2} + x[3] W_4^{3*3}$$

$$X[0] = x[0] W_4^0 + x[1] W_4^0 + x[2] W_4^0 + x[3] W_4^0$$

$$X[1] = x[0] W_4^0 + x[1] W_4^1 + x[2] W_4^2 + x[3] W_4^3$$

$$X[2] = x[0] W_4^0 + x[1] W_4^2 + x[2] W_4^4 + x[3] W_4^6$$

$$X[3] = x[0] W_4^0 + x[1] W_4^3 + x[2] W_4^6 + x[3] W_4^9$$

$$\blacksquare W_4^k = e^{-i \frac{2\pi}{4} k}$$

$$W_4^0 = e^{-i \frac{2\pi}{4} 0} = \cos\left[-\frac{2\pi}{4} 0\right] + i \sin\left[-\frac{2\pi}{4} 0\right] = 1$$

$$W_4^1 = e^{-i \frac{2\pi}{4} 1} = \cos\left[-\frac{2\pi}{4} 1\right] + i \sin\left[-\frac{2\pi}{4} 1\right] = -i$$

$$W_4^2 = e^{-i \frac{2\pi}{4} 2} = \cos\left[-\frac{2\pi}{4} 2\right] + i \sin\left[-\frac{2\pi}{4} 2\right] = -1 = -W_4^0$$

$$W_4^3 = e^{-i \frac{2\pi}{4} 3} = \cos\left[-\frac{2\pi}{4} 3\right] + i \sin\left[-\frac{2\pi}{4} 3\right] = i = -W_4^1$$

$$W_4^4 = e^{-i \frac{2\pi}{4} 4} = \cos\left[-\frac{2\pi}{4} 4\right] + i \sin\left[-\frac{2\pi}{4} 4\right] = 1 = W_4^0$$

$$W_4^5 = e^{-i \frac{2\pi}{4} 5} = \cos\left[-\frac{2\pi}{4} 5\right] + i \sin\left[-\frac{2\pi}{4} 5\right] = -i = W_4^1$$

$$W_4^6 = e^{-i \frac{2\pi}{4} 6} = \cos\left[-\frac{2\pi}{4} 6\right] + i \sin\left[-\frac{2\pi}{4} 6\right] = -1 = -W_4^0$$

$$W_4^7 = e^{-i \frac{2\pi}{4} 7} = \cos\left[-\frac{2\pi}{4} 7\right] + i \sin\left[-\frac{2\pi}{4} 7\right] = i = -W_4^1$$

$$W_4^8 = e^{-i \frac{2\pi}{4} 8} = \cos\left[-\frac{2\pi}{4} 8\right] + i \sin\left[-\frac{2\pi}{4} 8\right] = 1 = W_4^0$$

$$W_4^9 = e^{-i \frac{2\pi}{4} 9} = \cos\left[-\frac{2\pi}{4} 9\right] + i \sin\left[-\frac{2\pi}{4} 9\right] = -i = W_4^1$$

$$X[0] = x[0] W_4^0 + x[1] W_4^0 + x[2] W_4^0 + x[3] W_4^0$$

$$X[1] = x[0] W_4^0 + x[1] W_4^1 + x[2] W_4^2 + x[3] W_4^3$$

$$X[2] = x[0] W_4^0 + x[1] W_4^2 + x[2] W_4^0 + x[3] W_4^2$$

$$X[3] = x[0] W_4^0 + x[1] W_4^3 + x[2] W_4^2 + x[3] W_4^1$$

■ N=8

$$X[k] = \sum_{n=0}^{N-1} \left(x[n] e^{-j 2 \pi \frac{k}{N} n} \right) = \sum_{n=0}^{N-1} (x[n] W_N^{k n})$$

$$X[0] = x[0] W_8^{0*0} + x[1] W_8^{0*1} + x[2] W_8^{0*2} + x[3] W_8^{0*3} + x[4] W_8^{0*4} + x[5] W_8^{0*5} + x[6] W_8^{0*6} + x[7] W_8^{0*7}$$

$$X[1] = x[0] W_8^{1*0} + x[1] W_8^{1*1} + x[2] W_8^{1*2} + x[3] W_8^{1*3} + x[4] W_8^{1*4} + x[5] W_8^{1*5} + x[6] W_8^{1*6} + x[7] W_8^{1*7}$$

$$X[2] = x[0] W_8^{2*0} + x[1] W_8^{2*1} + x[2] W_8^{2*2} + x[3] W_8^{2*3} + x[4] W_8^{2*4} + x[5] W_8^{2*5} + x[6] W_8^{2*6} + x[7] W_8^{2*7}$$

$$X[3] = x[0] W_8^{3*0} + x[1] W_8^{3*1} + x[2] W_8^{3*2} + x[3] W_8^{3*3} + x[4] W_8^{3*4} + x[5] W_8^{3*5} + x[6] W_8^{3*6} + x[7] W_8^{3*7}$$

$$X[4] = x[0] W_8^{4*0} + x[1] W_8^{4*1} + x[2] W_8^{4*2} + x[3] W_8^{4*3} + x[4] W_8^{4*4} + x[5] W_8^{4*5} + x[6] W_8^{4*6} + x[7] W_8^{4*7}$$

$$X[5] = x[0] W_8^{5*0} + x[1] W_8^{5*1} + x[2] W_8^{5*2} + x[3] W_8^{5*3} + x[4] W_8^{5*4} + x[5] W_8^{5*5} + x[6] W_8^{5*6} + x[7] W_8^{5*7}$$

$$X[6] = x[0] W_8^{6*0} + x[1] W_8^{6*1} + x[2] W_8^{6*2} + x[3] W_8^{6*3} + x[4] W_8^{6*4} + x[5] W_8^{6*5} + x[6] W_8^{6*6} + x[7] W_8^{6*7}$$

$$X[7] = x[0] W_8^{7*0} + x[1] W_8^{7*1} + x[2] W_8^{7*2} + x[3] W_8^{7*3} + x[4] W_8^{7*4} + x[5] W_8^{7*5} + x[6] W_8^{7*6} + x[7] W_8^{7*7}$$

$$X[0] = x[0] W_8^0 + x[1] W_8^0 + x[2] W_8^0 + x[3] W_8^0 + x[4] W_8^0 + x[5] W_8^0 + x[6] W_8^0 + x[7] W_8^0$$

$$X[1] = x[0] W_8^0 + x[1] W_8^1 + x[2] W_8^2 + x[3] W_8^3 + x[4] W_8^4 + x[5] W_8^5 + x[6] W_8^6 + x[7] W_8^7$$

$$X[2] = x[0] W_8^0 + x[1] W_8^2 + x[2] W_8^4 + x[3] W_8^6 + x[4] W_8^8 + x[5] W_8^{10} + x[6] W_8^{12} + x[7] W_8^{14}$$

$$X[3] = x[0] W_8^0 + x[1] W_8^3 + x[2] W_8^6 + x[3] W_8^9 + x[4] W_8^{12} + x[5] W_8^{15} + x[6] W_8^{18} + x[7] W_8^{21}$$

$$X[4] = x[0] W_8^0 + x[1] W_8^4 + x[2] W_8^8 + x[3] W_8^{12} + x[4] W_8^{16} + x[5] W_8^{20} + x[6] W_8^{24} + x[7] W_8^{28}$$

$$X[5] = x[0] W_8^0 + x[1] W_8^5 + x[2] W_8^{10} + x[3] W_8^{15} + x[4] W_8^{20} + x[5] W_8^{25} + x[6] W_8^{30} + x[7] W_8^{35}$$

$$X[6] = x[0] W_8^0 + x[1] W_8^6 + x[2] W_8^{12} + x[3] W_8^{18} + x[4] W_8^{24} + x[5] W_8^{30} + x[6] W_8^{36} + x[7] W_8^{42}$$

$$X[7] = x[0] W_8^0 + x[1] W_8^7 + x[2] W_8^{14} + x[3] W_8^{21} + x[4] W_8^{28} + x[5] W_8^{35} + x[6] W_8^{42} + x[7] W_8^{49}$$

$$X[0] = x[0] W_8^0 + x[1] W_8^0 + x[2] W_8^0 + x[3] W_8^0 + x[4] W_8^0 + x[5] W_8^0 + x[6] W_8^0 + x[7] W_8^0$$

$$X[1] = x[0] W_8^0 + x[1] W_8^1 + x[2] W_8^2 + x[3] W_8^3 + x[4] W_8^4 + x[5] W_8^5 + x[6] W_8^6 + x[7] W_8^7$$

$$X[2] = x[0] W_8^0 + x[1] W_8^2 + x[2] W_8^4 + x[3] W_8^6 + x[4] W_8^8 + x[5] W_8^{10} + x[6] W_8^{12} + x[7] W_8^{14}$$

$$X[3] = x[0] W_8^0 + x[1] W_8^3 + x[2] W_8^6 + x[3] W_8^9 + x[4] W_8^{12} + x[5] W_8^{15} + x[6] W_8^{18} + x[7] W_8^{21}$$

$$X[4] = x[0] W_8^0 + x[1] W_8^4 + x[2] W_8^8 + x[3] W_8^{12} + x[4] W_8^{16} + x[5] W_8^{20} + x[6] W_8^{24} + x[7] W_8^{28}$$

$$X[5] = x[0] W_8^0 + x[1] W_8^5 + x[2] W_8^2 + x[3] W_8^7 + x[4] W_8^4 + x[5] W_8^1 + x[6] W_8^6 + x[7] W_8^3$$

$$X[6] = x[0] W_8^0 + x[1] W_8^6 + x[2] W_8^4 + x[3] W_8^2 + x[4] W_8^0 + x[5] W_8^6 + x[6] W_8^4 + x[7] W_8^2$$

$$X[7] = x[0] W_8^0 + x[1] W_8^7 + x[2] W_8^6 + x[3] W_8^5 + x[4] W_8^4 + x[5] W_8^3 + x[6] W_8^2 + x[7] W_8^1$$

■ $W_8^k = e^{-j \frac{2\pi}{8} k}$

$$W_8^0 = e^{-j \frac{2\pi}{8} 0} = \cos\left[-\frac{2\pi}{8} 0\right] + j \sin\left[-\frac{2\pi}{8} 0\right] = 1$$

$$W_8^1 = e^{-j \frac{2\pi}{8} 1} = \cos\left[-\frac{2\pi}{8} 1\right] + j \sin\left[-\frac{2\pi}{8} 1\right] = \frac{1-j}{\sqrt{2}}$$

$$W_8^2 = e^{-j \frac{2\pi}{8} 2} = \cos\left[-\frac{2\pi}{8} 2\right] + j \sin\left[-\frac{2\pi}{8} 2\right] = -j$$

$$W_8^3 = e^{-j \frac{2\pi}{8} 3} = \cos\left[-\frac{2\pi}{8} 3\right] + j \sin\left[-\frac{2\pi}{8} 3\right] = -\frac{1+j}{\sqrt{2}}$$

$$W_8^4 = e^{-j \frac{2\pi}{8} 4} = \cos\left[-\frac{2\pi}{8} 4\right] + j \sin\left[-\frac{2\pi}{8} 4\right] = -1 = -W_8^0$$

$$W_8^5 = e^{-j \frac{2\pi}{8} 5} = \cos\left[-\frac{2\pi}{8} 5\right] + j \sin\left[-\frac{2\pi}{8} 5\right] = -\frac{1-j}{\sqrt{2}} = -W_8^1$$

$$W_8^6 = e^{-j \frac{2\pi}{8} 6} = \cos\left[-\frac{2\pi}{8} 6\right] + j \sin\left[-\frac{2\pi}{8} 6\right] = j = -W_8^2$$

$$W_8^7 = e^{-j \frac{2\pi}{8} 7} = \cos\left[-\frac{2\pi}{8} 7\right] + j \sin\left[-\frac{2\pi}{8} 7\right] = \frac{1+j}{\sqrt{2}} = -W_8^3$$

$$X[0] = x[0] W_8^0 + x[1] W_8^0 + x[2] W_8^0 + x[3] W_8^0 + x[4] W_8^0 + x[5] W_8^0 + x[6] W_8^0 + x[7] W_8^0$$

$$X[1] = x[0] W_8^0 + x[1] W_8^1 + x[2] W_8^2 + x[3] W_8^3 + x[4] (-W_8^0) + x[5] (-W_8^1) + x[6] (-W_8^2) + x[7] (-W_8^3)$$

$$X[2] = x[0] W_8^0 + x[1] W_8^2 + x[2] (-W_8^0) + x[3] (-W_8^2) + x[4] W_8^0 + x[5] W_8^2 + x[6] (-W_8^0) + x[7] (-W_8^2)$$

$$X[3] = x[0] W_8^0 + x[1] W_8^3 + x[2] (-W_8^2) + x[3] W_8^1 + x[4] (-W_8^0) + x[5] (-W_8^3) + x[6] W_8^2 + x[7] (-W_8^1)$$

$$X[4] = x[0] W_8^0 + x[1] (-W_8^0) + x[2] W_8^0 + x[3] (-W_8^0) + x[4] W_8^0 + x[5] (-W_8^0) + x[6] W_8^0 + x[7] (-W_8^0)$$

$$X[5] = x[0] W_8^0 + x[1] (-W_8^1) + x[2] W_8^2 + x[3] (-W_8^3) + x[4] (-W_8^0) + x[5] W_8^1 + x[6] (-W_8^2) + x[7] W_8^3$$

$$X[6] = x[0] W_8^0 + x[1] (-W_8^2) + x[2] (-W_8^0) + x[3] W_8^2 + x[4] W_8^0 + x[5] (-W_8^2) + x[6] (-W_8^0) + x[7] W_8^2$$

$$X[7] = x[0] W_8^0 + x[1] (-W_8^3) + x[2] (-W_8^2) + x[3] (-W_8^1) + x[4] (-W_8^0) + x[5] W_8^3 + x[6] W_8^2 + x[7] W_8^1$$

$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{pmatrix} = \begin{pmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & -W_8^0 & -W_8^1 & -W_8^2 & -W_8^3 \\ W_8^0 & W_8^2 & -W_8^0 & -W_8^2 & W_8^0 & W_8^2 & -W_8^0 & -W_8^2 \\ W_8^0 & W_8^3 & -W_8^2 & W_8^1 & -W_8^0 & -W_8^3 & W_8^2 & -W_8^1 \\ W_8^0 & -W_8^0 & W_8^0 & -W_8^0 & W_8^0 & -W_8^0 & W_8^0 & -W_8^0 \\ W_8^0 & -W_8^1 & W_8^2 & -W_8^3 & -W_8^0 & W_8^1 & -W_8^2 & W_8^3 \\ W_8^0 & -W_8^2 & -W_8^0 & W_8^2 & W_8^0 & -W_8^2 & -W_8^0 & W_8^2 \\ W_8^0 & -W_8^3 & -W_8^2 & -W_8^1 & -W_8^0 & W_8^3 & W_8^2 & W_8^1 \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{pmatrix}$$

■ 8-Point FFT

$$W_N^k = e^{-j \frac{2\pi k}{N}}$$

$$\begin{pmatrix} 1 & W_2^0 \\ 1 & -W_2^0 \end{pmatrix} \cdot \begin{pmatrix} x[0] & x[2] \\ x[4] & x[6] \end{pmatrix}$$

$$\begin{pmatrix} 1 & W_2^0 \\ 1 & -W_2^0 \end{pmatrix} \cdot \begin{pmatrix} x[1] & x[3] \\ x[5] & x[7] \end{pmatrix}$$

$$\begin{pmatrix} Y_{20} & Z_{20} \\ Y_{21} & Z_{21} \end{pmatrix} = \begin{pmatrix} 1 & W_2^0 \\ 1 & -W_2^0 \end{pmatrix} \cdot \begin{pmatrix} x[0] & x[2] \\ x[4] & x[6] \end{pmatrix}$$

$$\begin{pmatrix} Y_{20} \\ Y_{21} \end{pmatrix} = \begin{pmatrix} 1 & W_2^0 \\ 1 & -W_2^0 \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[4] \end{pmatrix}$$

$$\begin{pmatrix} Y_{40} \\ Y_{42} \end{pmatrix} = \begin{pmatrix} 1 & W_4^0 \\ 1 & -W_4^0 \end{pmatrix} \cdot \begin{pmatrix} Y_{20} \\ Z_{20} \end{pmatrix}$$

$$\begin{pmatrix} Y_{41} \\ Y_{43} \end{pmatrix} = \begin{pmatrix} 1 & W_4^1 \\ 1 & -W_4^1 \end{pmatrix} \cdot \begin{pmatrix} Y_{21} \\ Z_{21} \end{pmatrix}$$

$$\begin{pmatrix} Y_{40} \\ Y_{41} \\ Y_{42} \\ Y_{43} \end{pmatrix} = \begin{pmatrix} 1 & 0 & W_4^0 & 0 \\ 0 & 1 & 0 & W_4^1 \\ 1 & 0 & -W_4^0 & 0 \\ 0 & 1 & 0 & -W_4^1 \end{pmatrix} \cdot \begin{pmatrix} Y_{20} \\ Y_{21} \\ Z_{20} \\ Z_{21} \end{pmatrix}$$

$$\begin{pmatrix} Y_{40} \\ Y_{41} \\ Y_{42} \\ Y_{43} \\ Z_{40} \\ Z_{41} \\ Z_{42} \\ Z_{43} \end{pmatrix} = \begin{pmatrix} 1 & 0 & W_4^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & W_4^1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -W_4^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -W_4^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & W_4^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & W_4^1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -W_4^0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -W_4^1 \end{pmatrix} \cdot \begin{pmatrix} Y_{20} \\ Y_{21} \\ Z_{20} \\ Z_{21} \\ \square \\ \square \\ \square \\ \square \end{pmatrix}$$

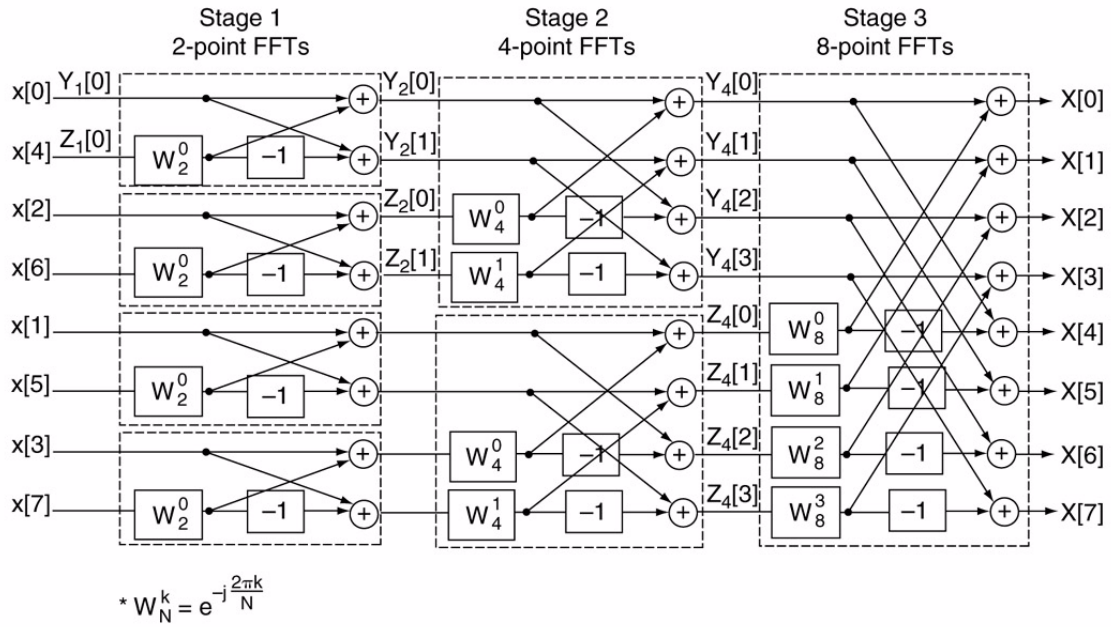
$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & W_8^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_8^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^3 \\ 1 & 0 & 0 & 0 & -W_8^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -W_8^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -W_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -W_8^3 \end{pmatrix} \cdot \begin{pmatrix} Y_{40} \\ Y_{41} \\ Y_{42} \\ Y_{43} \\ Z_{40} \\ Z_{41} \\ Z_{42} \\ Z_{43} \end{pmatrix}$$

■ Summary

$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & W_8^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_8^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^3 \\ 1 & 0 & 0 & 0 & -W_8^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -W_8^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -W_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -W_8^3 \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} 1 & 0 & W_4^0 & 0 \\ 0 & 1 & 0 & W_4^1 \\ 1 & 0 & -W_4^0 & 0 \\ 0 & 1 & 0 & -W_4^1 \end{pmatrix} \cdot \begin{pmatrix} (1 \ W_2^0) \cdot (x[0]) \\ (1 \ -W_2^0) \cdot (x[4]) \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & W_4^0 & 0 \\ 0 & 1 & 0 & W_4^1 \\ 1 & 0 & -W_4^0 & 0 \\ 0 & 1 & 0 & -W_4^1 \end{pmatrix} \cdot \begin{pmatrix} (1 \ W_2^0) \cdot (x[1]) \\ (1 \ -W_2^0) \cdot (x[5]) \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & W_4^0 & 0 \\ 0 & 1 & 0 & W_4^1 \\ 1 & 0 & -W_4^0 & 0 \\ 0 & 1 & 0 & -W_4^1 \end{pmatrix} \cdot \begin{pmatrix} (1 \ W_2^0) \cdot (x[2]) \\ (1 \ -W_2^0) \cdot (x[6]) \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & W_4^0 & 0 \\ 0 & 1 & 0 & W_4^1 \\ 1 & 0 & -W_4^0 & 0 \\ 0 & 1 & 0 & -W_4^1 \end{pmatrix} \cdot \begin{pmatrix} (1 \ W_2^0) \cdot (x[3]) \\ (1 \ -W_2^0) \cdot (x[7]) \end{pmatrix} \end{pmatrix}$$

$$W_2^0 = W_4^0 = W_8^0$$

$$W_4^1 = W_8^2$$



■ s1

$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & W_8^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_8^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^3 \\ 1 & 0 & 0 & 0 & -W_8^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -W_8^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -W_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -W_8^3 \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} 1 & 0 & W_4^0 & 0 \\ 0 & 1 & 0 & W_4^1 \\ 1 & 0 & -W_4^0 & 0 \\ 0 & 1 & 0 & -W_4^1 \end{pmatrix} \cdot \begin{pmatrix} 1 & W_2^0 & 0 & 0 \\ 1 & -W_2^0 & 0 & 0 \\ 0 & 0 & 1 & W_2^0 \\ 0 & 0 & 1 & -W_2^0 \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[4] \\ x[2] \\ x[6] \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & W_4^0 & 0 \\ 0 & 1 & 0 & W_4^1 \\ 1 & 0 & -W_4^0 & 0 \\ 0 & 1 & 0 & -W_4^1 \end{pmatrix} \cdot \begin{pmatrix} 1 & W_2^0 & 0 & 0 \\ 1 & -W_2^0 & 0 & 0 \\ 0 & 0 & 1 & W_2^0 \\ 0 & 0 & 1 & -W_2^0 \end{pmatrix} \cdot \begin{pmatrix} x[1] \\ x[5] \\ x[3] \\ x[7] \end{pmatrix} \end{pmatrix}$$

■ t2

$$\begin{pmatrix} z[0] \\ z[1] \\ z[2] \\ z[3] \\ z[4] \\ z[5] \\ z[6] \\ z[7] \end{pmatrix} = \begin{pmatrix} 1 & 0 & W_4^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & W_4^1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -W_4^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -W_4^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & W_4^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & W_4^1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -W_4^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -W_4^1 \end{pmatrix} \cdot \begin{pmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \\ y[7] \end{pmatrix}$$

$$\begin{pmatrix} z[0] \\ z[1] \\ z[2] \\ z[3] \\ z[4] \\ z[5] \\ z[6] \\ z[7] \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & W_4^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_4^1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -W_4^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -W_4^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_4^0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_4^1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -W_4^0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -W_4^1 \end{pmatrix} \cdot \begin{pmatrix} y[0] \\ y[1] \\ y[4] \\ y[5] \\ y[2] \\ y[3] \\ y[6] \\ y[7] \end{pmatrix}$$

$$\begin{pmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \\ y[7] \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & W_2^0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -W_2^0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_2^0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -W_2^0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_2^0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -W_2^0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_2^0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -W_2^0 \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{pmatrix}$$

$$\begin{pmatrix} y[0] \\ y[1] \\ y[4] \\ y[5] \\ y[2] \\ y[3] \\ y[6] \\ y[7] \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & W_2^0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -W_2^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_2^0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -W_2^0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_2^0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -W_2^0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_2^0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -W_2^0 \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{pmatrix}$$

■ Good

$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & W_8^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_8^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^3 \\ 1 & 0 & 0 & 0 & -W_8^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -W_8^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -W_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -W_8^3 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & W_4^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_4^1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -W_4^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -W_4^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_4^0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_4^1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -W_4^0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -W_4^1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & W_2^0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -W_2^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_2^0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -W_2^0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_2^0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -W_2^0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_2^0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -W_2^0 \end{pmatrix} \cdot \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{pmatrix}$$

■ Cooley Turkey Algorithm

$$f_j = \sum_{k=0}^{n-1} x_k W^{jk}$$

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^1 & W^2 & W^3 \\ W^0 & W^2 & W^4 & W^6 \\ W^0 & W^3 & W^6 & W^9 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^2 & W^1 & W^3 \\ W^0 & W^4 & W^2 & W^6 \\ W^0 & W^6 & W^3 & W^9 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^2 & W^1 & W^3 \\ W^0 & W^4 & W^2 & W^6 \\ W^0 & W^6 & W^3 & W^9 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & W^0 & 0 \\ 0 & 1 & 0 & W^1 \\ 1 & 0 & W^2 & 0 \\ 0 & 1 & 0 & W^3 \end{pmatrix} \cdot \begin{pmatrix} \begin{pmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \end{pmatrix}$$

$$\blacksquare e^{it} = \cos[t] + i \sin[t]$$

$$e^{it} = \cos[t] + i \sin[t]$$

$$e^{i2\pi t} = \cos[2\pi t] + i \sin[2\pi t]$$

$$\begin{aligned}
 e^{-i 2 \pi t} &= \text{Cos}[-2 \pi t] + i \text{Sin}[-2 \pi t] \\
 &= \text{Cos}[2 \pi t] - i \text{Sin}[2 \pi t]
 \end{aligned}$$

Cos[α], Sin[α]

$$\text{Cos}\left[\alpha - \frac{3}{2} \pi\right] = -\text{Sin}[\alpha]$$

$$\text{Sin}\left[\alpha - \frac{3}{2} \pi\right] = \text{Cos}[\alpha]$$

$$\text{Cos}[\alpha - \pi] = -\text{Cos}[\alpha]$$

$$\text{Sin}[\alpha - \pi] = -\text{Sin}[\alpha]$$

$$\text{Cos}\left[\alpha - \frac{\pi}{2}\right] = \text{Sin}[\alpha]$$

$$\text{Sin}\left[\alpha - \frac{\pi}{2}\right] = -\text{Cos}[\alpha]$$

$$\text{Cos}\left[\alpha + \frac{\pi}{2}\right] = -\text{Sin}[\alpha]$$

$$\text{Sin}\left[\alpha + \frac{\pi}{2}\right] = \text{Cos}[\alpha]$$

$$\text{Cos}[\alpha + \pi] = -\text{Cos}[\alpha]$$

$$\text{Sin}[\alpha + \pi] = -\text{Sin}[\alpha]$$

$$\text{Cos}\left[\alpha + \frac{3}{2} \pi\right] = \text{Sin}[\alpha]$$

$$\text{Sin}\left[\alpha + \frac{3}{2} \pi\right] = -\text{Cos}[\alpha]$$

$$\text{Cos}[\alpha + 2 \pi] = \text{Cos}[\alpha]$$

$$\text{Sin}[\alpha + 2 \pi] = \text{Sin}[\alpha]$$

$$\text{Cos}[-\alpha] = \text{Cos}[\alpha]$$

$$\text{Sin}[-\alpha] = -\text{Sin}[\alpha]$$

$$e^{i t} = \text{Cos}[t] + i \text{Sin}[t]$$

Log

$$\log_a N = m \iff a^m = N$$

$$\log_a a = 1$$

$$\log_a 1 = 0$$

$$\log_a MN = \log_a M + \log_a N$$

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

$$\log_a M^n = n \log_a M$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

$$\log_a b = \frac{1}{\log_b a}$$

$$\frac{\log_c a}{\log_c b} = \log_b a = c$$

$$b^c = a$$

1D

$$X_N^{kn} = e^{-j 2 \pi \frac{kn}{N}}$$

$$n = 0, \dots, N-1$$

$$k = 0, \dots, N-1$$

$$S[k] = \frac{1}{N} \sum_{n=0}^{N-1} s[n] e^{-j 2 \pi \frac{kn}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} s[n] X_N^{kn}$$

$$s[n] = \frac{1}{N} \sum_{k=0}^{N-1} S[k] e^{j 2 \pi \frac{kn}{N}} = \frac{1}{N} \sum_{k=0}^{N-1} S[k] X_N^{-kn}$$

$$N = 2^n$$

$$N = 2L$$

$$\begin{aligned} S[k] &= \frac{1}{2L} \sum_{n=0}^{2L-1} s[n] e^{-j 2 \pi \frac{kn}{2L}} = \frac{1}{2L} \sum_{n=0}^{2L-1} s[n] X_{2L}^{kn} \\ &= \frac{1}{2} \left(\frac{1}{L} \sum_{n=0}^{L-1} s[2n] X_{2L}^{k(2n)} + \frac{1}{L} \sum_{n=0}^{L-1} s[2n+1] X_{2L}^{k(2n+1)} \right) \end{aligned}$$

$$X_{2L}^{k(2n)} = e^{-j 2 \pi \frac{k(2n)}{2L}} = e^{-j 2 \pi \frac{kn}{L}} = X_L^{kn}$$

$$X_{2L}^{k(2n+1)} = X_{2L}^{k(2n)} X_{2L}^k = X_L^{kn} X_{2L}^k$$

$$S_{\text{even}}[k] = \frac{1}{L} \sum_{n=0}^{L-1} s[2n] X_L^{kn}$$

$$S_{\text{odd}}[k] = \frac{1}{L} \sum_{n=0}^{L-1} s[2n+1] X_L^{kn}$$

$$S[k] = \frac{1}{2} \left(\frac{1}{L} \sum_{n=0}^{L-1} s[2n] X_L^{kn} + \frac{1}{L} \sum_{n=0}^{L-1} s[2n+1] X_L^{kn} X_{2L}^k \right)$$

$$S[k] = \frac{1}{2} (S_{\text{even}}[k] + S_{\text{odd}}[k] X_{2L}^k)$$

$$\begin{aligned}
S[k+L] &= \frac{1}{2} \left(\frac{1}{L} \sum_{n=0}^{L-1} s[2n] X_L^{(k+L)n} + \frac{1}{L} \sum_{n=0}^{L-1} s[2n+1] X_L^{(k+L)n} X_{2L}^{k+L} \right) \\
&= \frac{1}{2} (S_{\text{even}}[k+L] + S_{\text{odd}}[k+L] X_{2L}^{k+L}) \\
&= \frac{1}{2} (S_{\text{even}}[k] - S_{\text{odd}}[k] X_{2L}^k) \\
X_L^{k+L} &= e^{-j 2\pi \frac{k+L}{L}} = e^{-j 2\pi \frac{k}{L}} e^{-j 2\pi} = e^{-j 2\pi \frac{k}{L}} = X_L^k \\
X_{2L}^{k+L} &= e^{-j 2\pi \frac{k+L}{2L}} = e^{-j 2\pi \frac{k}{2L}} e^{-j 2\pi \frac{1}{2}} = -e^{-j 2\pi \frac{k}{2L}} = -X_{2L}^k
\end{aligned}$$

2D

$$\begin{aligned}
X_N^{kn} &= e^{-j 2\pi \frac{kn}{N}} \\
S[u, v] &= \frac{1}{N} \sum_{r=0}^{N-1} \sum_{c=0}^{N-1} s[r, c] e^{-j 2\pi \frac{cu}{N}} e^{-j 2\pi \frac{rv}{N}} = \frac{1}{N} \sum_{r=0}^{N-1} \sum_{c=0}^{N-1} s[r, c] e^{-j 2\pi \left(\frac{cu}{N} + \frac{rv}{N} \right)} = \frac{1}{N} \sum_{r=0}^{N-1} \sum_{c=0}^{N-1} s[r, c] X_N^{(cu+rv)} \\
s[r, c] &= \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} S[u, v] e^{j 2\pi \frac{cu}{N}} e^{j 2\pi \frac{rv}{N}} = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} S[u, v] e^{j 2\pi \left(\frac{cu}{N} + \frac{rv}{N} \right)} = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} S[u, v] X_N^{-(cu+rv)}
\end{aligned}$$