

DCT (Discrete Cosine Transform)

$$a_{ij} = \begin{cases} \sqrt{\frac{1}{n}} & i = 0 \\ \sqrt{\frac{2}{n}} \cos\left[\frac{(2j+1)i\pi}{2n}\right] & i > 0 \end{cases}$$

■ N=2

■ orthonormal (compaction)

■ $A = A^{-1} = A^T$

$n = 2;$

$$A = \begin{pmatrix} \sqrt{\frac{1}{n}} & \sqrt{\frac{1}{n}} \\ \sqrt{\frac{2}{n}} \cos\left[\frac{11\pi}{2n}\right] & \sqrt{\frac{2}{n}} \cos\left[\frac{31\pi}{2n}\right] \end{pmatrix};$$

A // N // MatrixForm

$$\begin{pmatrix} 0.707107 & 0.707107 \\ 0.707107 & -0.707107 \end{pmatrix}$$

Inverse[A] // N // MatrixForm

$$\begin{pmatrix} 0.707107 & 0.707107 \\ 0.707107 & -0.707107 \end{pmatrix}$$

Transpose[A] // N // MatrixForm

$$\begin{pmatrix} 0.707107 & 0.707107 \\ 0.707107 & -0.707107 \end{pmatrix}$$

■ N=4

$n = 4;$

$$A = \begin{pmatrix} \sqrt{\frac{1}{n}} & \sqrt{\frac{1}{n}} & \sqrt{\frac{1}{n}} & \sqrt{\frac{1}{n}} \\ \sqrt{\frac{2}{n}} \cos\left[\frac{11\pi}{2n}\right] & \sqrt{\frac{2}{n}} \cos\left[\frac{31\pi}{2n}\right] & \sqrt{\frac{2}{n}} \cos\left[\frac{51\pi}{2n}\right] & \sqrt{\frac{2}{n}} \cos\left[\frac{71\pi}{2n}\right] \\ \sqrt{\frac{2}{n}} \cos\left[\frac{12\pi}{2n}\right] & \sqrt{\frac{2}{n}} \cos\left[\frac{32\pi}{2n}\right] & \sqrt{\frac{2}{n}} \cos\left[\frac{52\pi}{2n}\right] & \sqrt{\frac{2}{n}} \cos\left[\frac{72\pi}{2n}\right] \\ \sqrt{\frac{2}{n}} \cos\left[\frac{13\pi}{2n}\right] & \sqrt{\frac{2}{n}} \cos\left[\frac{33\pi}{2n}\right] & \sqrt{\frac{2}{n}} \cos\left[\frac{53\pi}{2n}\right] & \sqrt{\frac{2}{n}} \cos\left[\frac{73\pi}{2n}\right] \end{pmatrix};$$

A // N // MatrixForm

$$\begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.653281 & 0.270598 & -0.270598 & -0.653281 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ 0.270598 & -0.653281 & 0.653281 & -0.270598 \end{pmatrix}$$

Inverse[A] // N // MatrixForm

$$\begin{pmatrix} 0.5 & 0.653281 & 0.5 & 0.270598 \\ 0.5 & 0.270598 & -0.5 & -0.653281 \\ 0.5 & -0.270598 & -0.5 & 0.653281 \\ 0.5 & -0.653281 & 0.5 & -0.270598 \end{pmatrix}$$

Transpose[A] // N // MatrixForm

$$\begin{pmatrix} 0.5 & 0.653281 & 0.5 & 0.270598 \\ 0.5 & 0.270598 & -0.5 & -0.653281 \\ 0.5 & -0.270598 & -0.5 & 0.653281 \\ 0.5 & -0.653281 & 0.5 & -0.270598 \end{pmatrix}$$

Transform

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$X = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A X = C$$

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

2D DCT

■ DCT

$$F[u, v] = \frac{2}{N} C[u] C[v] \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f[x, y] \cos\left[\frac{(2x+1)u\pi}{2N}\right] \cos\left[\frac{(2y+1)v\pi}{2N}\right]$$

$$f[x, y] = \frac{2}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} C[u] C[v] F[u, v] \cos\left[\frac{(2x+1)u\pi}{2N}\right] \cos\left[\frac{(2y+1)v\pi}{2N}\right]$$

$$C[u], C[v] = \begin{cases} \frac{1}{\sqrt{2}} & u, v = 0 \\ 1 & \text{otherwise} \end{cases}$$

2-Dimensional Block Transform

■ Block of Pixels X

$$\begin{array}{cccc} x_{00} & x_{01} & x_{02} & x_{03} \\ x_{10} & x_{11} & x_{12} & x_{13} \\ x_{20} & x_{21} & x_{22} & x_{23} \\ x_{30} & x_{31} & x_{32} & x_{33} \end{array}$$

■ Transform

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$A^T = \begin{pmatrix} a_{00} & a_{10} & a_{20} & a_{30} \\ a_{01} & a_{11} & a_{21} & a_{31} \\ a_{02} & a_{12} & a_{22} & a_{32} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$\text{Transform rows } r_{ij} = \sum_{k=0}^{N-1} a_{jk} x_{ik}$$

$$\text{Transform columns } c_{ij} = \sum_{m=0}^{N-1} a_{im} r_{mj} = \sum_{m=0}^{N-1} a_{im} \sum_{k=0}^{N-1} a_{jk} x_{mk} = \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} a_{im} x_{mk} a_{jk}$$

$$\text{Summary : } C = A X A^T$$

$$X A \Rightarrow \sum_{k=0}^{N-1} x_{ik} a_{kj}$$

$$R = X A^T \Rightarrow r_{ij} = \sum_{k=0}^{N-1} x_{ik} a_{jk}$$

$$A R \Rightarrow c_{ij} = \sum_{m=0}^{N-1} a_{im} r_{mj}$$

$$A X A^T \Rightarrow c_{ij} = \sum_{m=0}^{N-1} a_{im} \sum_{k=0}^{N-1} x_{mk} a_{jk} = \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} a_{im} x_{mk} a_{jk}$$

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} x_{00} & x_{01} & x_{02} & x_{03} \\ x_{10} & x_{11} & x_{12} & x_{13} \\ x_{20} & x_{21} & x_{22} & x_{23} \\ x_{30} & x_{31} & x_{32} & x_{33} \end{pmatrix} \cdot \begin{pmatrix} a_{00} & a_{10} & a_{20} & a_{30} \\ a_{01} & a_{11} & a_{21} & a_{31} \\ a_{02} & a_{12} & a_{22} & a_{32} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix} // \text{MatrixForm}$$

■ Row Space

$$(x_{*0} \quad x_{*1} \quad x_{*2} \quad x_{*3})$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_{00} & \mathbf{a}_{01} & \mathbf{a}_{02} & \mathbf{a}_{03} \\ \mathbf{a}_{10} & \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ \mathbf{a}_{20} & \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} \\ \mathbf{a}_{30} & \mathbf{a}_{31} & \mathbf{a}_{32} & \mathbf{a}_{33} \end{pmatrix}$$

$$\mathbf{X}_{\text{rs}}^{\text{T}} = \begin{pmatrix} \mathbf{x}_{*0} \\ \mathbf{x}_{*1} \\ \mathbf{x}_{*2} \\ \mathbf{x}_{*3} \end{pmatrix}$$

$$\mathbf{A} \mathbf{X}_{\text{rs}}^{\text{T}} = \mathbf{C}$$

■ Column Space

$$\begin{pmatrix} \mathbf{x}_{0*} \\ \mathbf{x}_{1*} \\ \mathbf{x}_{2*} \\ \mathbf{x}_{3*} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_{00} & \mathbf{a}_{01} & \mathbf{a}_{02} & \mathbf{a}_{03} \\ \mathbf{a}_{10} & \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ \mathbf{a}_{20} & \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} \\ \mathbf{a}_{30} & \mathbf{a}_{31} & \mathbf{a}_{32} & \mathbf{a}_{33} \end{pmatrix}$$

$$\mathbf{X}_{\text{cs}} = \begin{pmatrix} \mathbf{x}_{0*} \\ \mathbf{x}_{1*} \\ \mathbf{x}_{2*} \\ \mathbf{x}_{3*} \end{pmatrix}$$

$$\mathbf{A} \mathbf{X}_{\text{cs}} = \mathbf{C}$$

■ Matrix

$$\begin{pmatrix} \mathbf{x}_{00} & \mathbf{x}_{01} & \mathbf{x}_{02} & \mathbf{x}_{03} \\ \mathbf{x}_{10} & \mathbf{x}_{11} & \mathbf{x}_{12} & \mathbf{x}_{13} \\ \mathbf{x}_{20} & \mathbf{x}_{21} & \mathbf{x}_{22} & \mathbf{x}_{23} \\ \mathbf{x}_{30} & \mathbf{x}_{31} & \mathbf{x}_{32} & \mathbf{x}_{33} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_{00} & \mathbf{a}_{01} & \mathbf{a}_{02} & \mathbf{a}_{03} \\ \mathbf{a}_{10} & \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ \mathbf{a}_{20} & \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} \\ \mathbf{a}_{30} & \mathbf{a}_{31} & \mathbf{a}_{32} & \mathbf{a}_{33} \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_{00} & \mathbf{x}_{01} & \mathbf{x}_{02} & \mathbf{x}_{03} \\ \mathbf{x}_{10} & \mathbf{x}_{11} & \mathbf{x}_{12} & \mathbf{x}_{13} \\ \mathbf{x}_{20} & \mathbf{x}_{21} & \mathbf{x}_{22} & \mathbf{x}_{23} \\ \mathbf{x}_{30} & \mathbf{x}_{31} & \mathbf{x}_{32} & \mathbf{x}_{33} \end{pmatrix}$$

$$\mathbf{C} = \mathbf{A} (\mathbf{A}^{\text{T}})^{\text{T}} = \mathbf{A} \mathbf{X} \mathbf{A}^{\text{T}}$$