

3D Vector

Direction

$$\begin{pmatrix} 1 & 0 & 0 & P0x \\ 0 & 1 & 0 & P0y \\ 0 & 0 & 1 & P0z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Zh] & -\sin[Zh] & 0 & 0 \\ \sin[Zh] & \cos[Zh] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Yh] & 0 & \sin[Yh] & 0 \\ 0 & 1 & 0 & 0 \\ -\sin[Yh] & 0 & \cos[Yh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[Xh] & -\sin[Xh] & 0 \\ 0 & \sin[Xh] & \cos[Xh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T[[1]] // \text{FullSimplify}$$

$$\begin{pmatrix} 1 & 0 & 0 & P0x \\ 0 & 1 & 0 & P0y \\ 0 & 0 & 1 & P0z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Zh] & -\sin[Zh] & 0 & 0 \\ \sin[Zh] & \cos[Zh] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Yh] & 0 & \sin[Yh] & 0 \\ 0 & 1 & 0 & 0 \\ -\sin[Yh] & 0 & \cos[Yh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[Xh] & -\sin[Xh] & 0 \\ 0 & \sin[Xh] & \cos[Xh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}^T[[1]] // \text{FullSimplify}$$

$$\begin{pmatrix} 1 & 0 & 0 & P0x \\ 0 & 1 & 0 & P0y \\ 0 & 0 & 1 & P0z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Zh] & -\sin[Zh] & 0 & 0 \\ \sin[Zh] & \cos[Zh] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Yh] & 0 & \sin[Yh] & 0 \\ 0 & 1 & 0 & 0 \\ -\sin[Yh] & 0 & \cos[Yh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[Xh] & -\sin[Xh] & 0 \\ 0 & \sin[Xh] & \cos[Xh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}^T[[1]] // \text{FullSimplify}$$

$$\{P0x + \cos[Yh] \cos[Zh], P0y + \cos[Yh] \sin[Zh], P0z - \sin[Yh], 1\}$$

$$\{P0x + \cos[Zh] \sin[Xh] \sin[Yh] - \cos[Xh] \sin[Zh], \\ P0y + \cos[Xh] \cos[Zh] + \sin[Xh] \sin[Yh] \sin[Zh], P0z + \cos[Yh] \sin[Xh], 1\}$$

$$\{P0x + \cos[Xh] \cos[Zh] \sin[Yh] + \sin[Xh] \sin[Zh], \\ P0y - \cos[Zh] \sin[Xh] + \cos[Xh] \sin[Yh] \sin[Zh], P0z + \cos[Xh] \cos[Yh], 1\}$$

Multiple

$$\begin{pmatrix} DX & 0 & 0 & POx \\ 0 & DX & 0 & POy \\ 0 & 0 & DX & POz \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Zh] & -\sin[Zh] & 0 & 0 \\ \sin[Zh] & \cos[Zh] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Yh] & 0 & \sin[Yh] & 0 \\ 0 & 1 & 0 & 0 \\ -\sin[Yh] & 0 & \cos[Yh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[Xh] & -\sin[Xh] & 0 \\ 0 & \sin[Xh] & \cos[Xh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T[[1]] // \text{FullSimplify}$$

$$\begin{pmatrix} DY & 0 & 0 & POx \\ 0 & DY & 0 & POy \\ 0 & 0 & DY & POz \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Zh] & -\sin[Zh] & 0 & 0 \\ \sin[Zh] & \cos[Zh] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Yh] & 0 & \sin[Yh] & 0 \\ 0 & 1 & 0 & 0 \\ -\sin[Yh] & 0 & \cos[Yh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[Xh] & -\sin[Xh] & 0 \\ 0 & \sin[Xh] & \cos[Xh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}^T[[1]] // \text{FullSimplify}$$

$$\begin{pmatrix} DZ & 0 & 0 & POx \\ 0 & DZ & 0 & POy \\ 0 & 0 & DZ & POz \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Zh] & -\sin[Zh] & 0 & 0 \\ \sin[Zh] & \cos[Zh] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Yh] & 0 & \sin[Yh] & 0 \\ 0 & 1 & 0 & 0 \\ -\sin[Yh] & 0 & \cos[Yh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[Xh] & -\sin[Xh] & 0 \\ 0 & \sin[Xh] & \cos[Xh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}^T[[1]] // \text{FullSimplify}$$

$$\{POx + DX \cos[Yh] \cos[Zh], POy + DX \cos[Yh] \sin[Zh], POz - DX \sin[Yh], 1\}$$

$$\{POx + DY \cos[Zh] \sin[Xh] \sin[Yh] - DY \cos[Xh] \sin[Zh], POy + DY \cos[Xh] \cos[Zh] + DY \sin[Xh] \sin[Yh] \sin[Zh], POz + DY \cos[Yh] \sin[Xh], 1\}$$

$$\{POx + DZ \cos[Xh] \cos[Zh] \sin[Yh] + DZ \sin[Xh] \sin[Zh], POy - DZ \cos[Zh] \sin[Xh] + DZ \cos[Xh] \sin[Yh] \sin[Zh], POz + DZ \cos[Xh] \cos[Yh], 1\}$$

$$\begin{pmatrix} 1 & 0 & 0 & P0x \\ 0 & 1 & 0 & P0y \\ 0 & 0 & 1 & P0z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Zh] & -\sin[Zh] & 0 & 0 \\ \sin[Zh] & \cos[Zh] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Yh] & 0 & \sin[Yh] & 0 \\ 0 & 1 & 0 & 0 \\ -\sin[Yh] & 0 & \cos[Yh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[Xh] & -\sin[Xh] & 0 \\ 0 & \sin[Xh] & \cos[Xh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} DX \\ 0 \\ 0 \\ 1 \end{pmatrix}^T[[1]] // \text{FullSimplify}$$

$$\begin{pmatrix} 1 & 0 & 0 & P0x \\ 0 & 1 & 0 & P0y \\ 0 & 0 & 1 & P0z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Zh] & -\sin[Zh] & 0 & 0 \\ \sin[Zh] & \cos[Zh] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Yh] & 0 & \sin[Yh] & 0 \\ 0 & 1 & 0 & 0 \\ -\sin[Yh] & 0 & \cos[Yh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[Xh] & -\sin[Xh] & 0 \\ 0 & \sin[Xh] & \cos[Xh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ DY \\ 0 \\ 1 \end{pmatrix}^T[[1]] // \text{FullSimplify}$$

$$\begin{pmatrix} 1 & 0 & 0 & P0x \\ 0 & 1 & 0 & P0y \\ 0 & 0 & 1 & P0z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Zh] & -\sin[Zh] & 0 & 0 \\ \sin[Zh] & \cos[Zh] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[Yh] & 0 & \sin[Yh] & 0 \\ 0 & 1 & 0 & 0 \\ -\sin[Yh] & 0 & \cos[Yh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos[Xh] & -\sin[Xh] & 0 \\ 0 & \sin[Xh] & \cos[Xh] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ DZ \\ 1 \end{pmatrix}^T[[1]] // \text{FullSimplify}$$

$$\{P0x + DX \cos[Yh] \cos[Zh], P0y + DX \cos[Yh] \sin[Zh], P0z - DX \sin[Yh], 1\}$$

$$\{P0x + DY \cos[Zh] \sin[Xh] \sin[Yh] - DY \cos[Xh] \sin[Zh], \\ P0y + DY \cos[Xh] \cos[Zh] + DY \sin[Xh] \sin[Yh] \sin[Zh], P0z + DY \cos[Yh] \sin[Xh], 1\}$$

$$\{P0x + DZ \cos[Xh] \cos[Zh] \sin[Yh] + DZ \sin[Xh] \sin[Zh], \\ P0y - DZ \cos[Zh] \sin[Xh] + DZ \cos[Xh] \sin[Yh] \sin[Zh], P0z + DZ \cos[Xh] \cos[Yh], 1\}$$

$$dX = \sqrt{PXx^2 + PXy^2}$$

$$Zh = \text{ArcTan}[PXx, PXy]$$

$$\tan[Zh] = \frac{PXy}{PXx} = \frac{\frac{PXy}{dX}}{\frac{PXx}{dX}} = \frac{\sin[Zh]}{\cos[Zh]}$$

$$dX = \sqrt{PXx^2 + PXy^2}$$

$$dX^2 = PXx^2 + PXy^2$$

$$dX = \frac{PXx^2 + PXy^2}{dX} = \frac{PXx^2}{dX} + \frac{PXy^2}{dX} = \frac{PXx}{dX} PXx + \frac{PXy}{dX} PXy \\ = \cos[Zh] PXx + \sin[Zh] PXy$$

$$DX = \sqrt{PXx^2 + PXy^2 + PXz^2}$$

$$Yh = \text{ArcTan}[\cos[Zh] PXx + \sin[Zh] PXy, -PXz]$$

$$\begin{aligned}
\tan[Y_h] &= \frac{-PX_z}{dX} = \frac{\frac{-PX_z}{dX}}{\frac{dX}{dX}} = \frac{\sin[Y_h]}{\cos[Y_h]} \\
DX &= \sqrt{dX^2 + (-PX_z)^2} \\
DX^2 &= dX^2 + (-PX_z)^2 \\
DX &= \frac{dX^2 + (-PX_z)^2}{DX} = \frac{dX^2}{DX} + \frac{(-PX_z)^2}{DX} = \frac{dX}{DX} dX + \frac{(-PX_z)}{DX} (-PX_z) \\
&= \cos[Y_h] dX + \sin[Y_h] (-PX_z) \\
&= \cos[Y_h] \left\{ \frac{PX_x^2 + PY_y^2}{dX} \right\} + \sin[Y_h] (-PX_z) \\
&= \cos[Y_h] \left\{ \frac{PX_x}{dX} PX_x + \frac{PY_y}{dX} PY_y \right\} + \sin[Y_h] (-PX_z) \\
&= \cos[Y_h] \{ \cos[Z_h] PX_x + \sin[Z_h] PY_y \} + \sin[Y_h] (-PX_z) \\
&= \cos[Y_h] \cos[Z_h] PX_x + \cos[Y_h] \sin[Z_h] PY_y + \sin[Y_h] (-PX_z) \\
&= \cos[Y_h] \cos[Z_h] PX_x + \cos[Y_h] \sin[Z_h] PY_y - \sin[Y_h] PX_z
\end{aligned}$$

Distance

$$\begin{aligned}
Z_h &= \text{ArcTan}[x, y] \\
Y_h &= \text{ArcTan}[\cos[Z_h] x + \sin[Z_h] y, z] \\
D &= \sqrt{d^2 + z^2} \\
D^2 &= d^2 + z^2 \\
D &= \frac{d^2 + z^2}{D} = \frac{d^2}{D} + \frac{z^2}{D} = \frac{d}{D} d + \frac{z}{D} z \\
&= \cos[Y_h] d + \sin[Y_h] z \\
&= \cos[Y_h] \left\{ \frac{x^2 + y^2}{d} \right\} + \sin[Y_h] z \\
&= \cos[Y_h] \left\{ \frac{x}{d} x + \frac{y}{d} y \right\} + \sin[Y_h] z \\
&= \cos[Y_h] \{ \cos[Z_h] x + \sin[Z_h] y \} + \sin[Y_h] z \\
&= \cos[Y_h] \cos[Z_h] x + \cos[Y_h] \sin[Z_h] y + \sin[Y_h] z
\end{aligned}$$

■ ∴

$$\{P0x + D \cos[Y_h] \cos[Z_h], P0y + D \cos[Y_h] \sin[Z_h], P0z - D \sin[Y_h]\}$$

$$\begin{aligned}
Z_h &= \text{ArcTan}[x, y] \\
Y_h &= \text{ArcTan}[\cos[Z_h] x + \sin[Z_h] y, z] \\
D &= \cos[Y_h] \cos[Z_h] x + \cos[Y_h] \sin[Z_h] y + \sin[Y_h] z
\end{aligned}$$

Ex. Formula and solve

■ 1

$$\{d \cos[\theta] \cos[\varphi], d \cos[\theta] \sin[\varphi], d \sin[\theta]\} = \{x, y, z\}$$

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d = (x - x0) / Cos[θ] Cos[φ]
d = (y - y0) / Cos[φ] Sin[θ]
d = (z - z0) / Sin[φ]

S = Simplify[
  Solve[{(x - x0) / Cos[θ] Cos[φ] == (y - y0) / Cos[φ] Sin[θ] == (z - z0) / Sin[φ], z == h},
    {x, y, z}][[1]]];
{x,
 y,
 z} /.
S

```

■ 2

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Simplify[Transpose[
  
$$\begin{pmatrix} \cos[\theta] & -\sin[\theta] & 0 \\ \sin[\theta] & \cos[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\varphi] & 0 & -\sin[\varphi] \\ 0 & 1 & 0 \\ \sin[\varphi] & 0 & \cos[\varphi] \end{pmatrix} \cdot \begin{pmatrix} d \\ d \tan[\phi] \cos[t] \\ d \tan[\phi] \sin[t] \end{pmatrix}][[1]] + \{x0, y0, z0\}$$

  {x0 + d (-Cos[t] Sin[θ] Tan[φ] + Cos[θ] (Cos[φ] - Sin[t] Sin[φ] Tan[φ])),
    y0 + d (Cos[φ] Sin[θ] + (Cos[t] Cos[θ] - Sin[t] Sin[θ] Sin[φ]) Tan[φ]),
    z0 + d (Sin[φ] + Cos[φ] Sin[t] Tan[φ])} = {x, y, z}

d = (x - x0) / (-Cos[t] Sin[θ] Tan[φ] + Cos[θ] (Cos[φ] - Sin[t] Sin[φ] Tan[φ]))
d = (y - y0) / (Cos[φ] Sin[θ] + (Cos[t] Cos[θ] - Sin[t] Sin[θ] Sin[φ]) Tan[φ])
d = (z - z0) / (Sin[φ] + Cos[φ] Sin[t] Tan[φ])

S = Solve[{(x - x0) / (-Cos[t] Sin[θ] Tan[φ] + Cos[θ] (Cos[φ] - Sin[t] Sin[φ] Tan[φ]))
  == (y - y0) / (Cos[φ] Sin[θ] + (Cos[t] Cos[θ] - Sin[t] Sin[θ] Sin[φ]) Tan[φ])
  == (z - z0) / (Sin[φ] + Cos[φ] Sin[t] Tan[φ]), y == h}, {x, y, z}][[1]] // Simplify //
  InputForm;
{x // InputForm, y // InputForm, z // InputForm} /. S[[1]]

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■ 3

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Simplify[Transpose[
  
$$\begin{pmatrix} \cos[\theta] & -\sin[\theta] & 0 \\ \sin[\theta] & \cos[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\varphi] & 0 & -\sin[\varphi] \\ 0 & 1 & 0 \\ \sin[\varphi] & 0 & \cos[\varphi] \end{pmatrix} \cdot \begin{pmatrix} d \\ d \tan[\phi] \\ 0 \end{pmatrix}][[1]] + \{x0, y0, z0\}$$

  {x0, y0, z0}

Simplify[
  Transpose[
    
$$\begin{pmatrix} \cos[\theta] & -\sin[\theta] & 0 \\ \sin[\theta] & \cos[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\varphi] & 0 & -\sin[\varphi] \\ 0 & 1 & 0 \\ \sin[\varphi] & 0 & \cos[\varphi] \end{pmatrix} \cdot \begin{pmatrix} D \\ 0 \\ D \tan[\phi] \end{pmatrix}][[1]] + \{x0, y0, z0\}$$

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■ 4

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{Cos[ThH] Cos[JhH], Cos[JhH] Sin[ThH], Sin[JhH]}

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Simplify[Transpose[
  
$$\begin{pmatrix} \cos[\theta] & -\sin[\theta] & 0 \\ \sin[\theta] & \cos[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos[\varphi] & 0 & -\sin[\varphi] \\ 0 & 1 & 0 \\ \sin[\varphi] & 0 & \cos[\varphi] \end{pmatrix} \cdot \begin{pmatrix} d \\ d \tan[\phi] \cos[t] \\ d \tan[\phi] \sin[t] \end{pmatrix}][[1]] + \{x0, y0, z0\} \\
\{x0 + d (-\cos[t] \sin[\theta] \tan[\phi] + \cos[\theta] (\cos[\varphi] - \sin[t] \sin[\varphi] \tan[\phi])), \\
y0 + d (\cos[\varphi] \sin[\theta] + (\cos[t] \cos[\theta] - \sin[t] \sin[\theta] \sin[\varphi]) \tan[\phi]), \\
z0 + d (\sin[\varphi] + \cos[\varphi] \sin[t] \tan[\phi])\} = \{x, y, z\} \\
d = (x - x0) / (-\cos[t] \sin[\theta] \tan[\phi] + \cos[\theta] (\cos[\varphi] - \sin[t] \sin[\varphi] \tan[\phi])) \\
d = (y - y0) / (\cos[\varphi] \sin[\theta] + (\cos[t] \cos[\theta] - \sin[t] \sin[\theta] \sin[\varphi]) \tan[\phi]) \\
d = (z - z0) / (\sin[\varphi] + \cos[\varphi] \sin[t] \tan[\phi]) \\
S = Solve[{(x - x0) / (-\cos[t] \sin[\theta] \tan[\phi] + \cos[\theta] (\cos[\varphi] - \sin[t] \sin[\varphi] \tan[\phi])) \\
== (y - y0) / (\cos[\varphi] \sin[\theta] + (\cos[t] \cos[\theta] - \sin[t] \sin[\theta] \sin[\varphi]) \tan[\phi]) \\
== (z - z0) / (\sin[\varphi] + \cos[\varphi] \sin[t] \tan[\phi])}, \\
Cos[ThH] Cos[JhH] (x - xh0) + Cos[JhH] Sin[ThH] (y - yh0) + Sin[JhH] (z - zh0) == 0}, \\
\{x, y, z\}][[1]] // Simplify // InputForm; \\
\{x // InputForm, y // InputForm, z // InputForm\} /. S[[1]]$$

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