

Euler's formula

$$e^{it} = \cos[t] + i \sin[t]$$

Bernoulli

$$\frac{1}{1+x^2} = \frac{1}{2} \left(\frac{1}{1-ix} + \frac{1}{1+ix} \right)$$

$$\int \frac{1}{1+ax} dx = \frac{1}{a} \ln(1+ax)$$

$$\ln(\cos[x] + i \sin[x]) = ix$$

$$z = x + iy = |z| (\cos[\phi] + i \sin[\phi]) = r e^{i\phi}$$

$$z = x - iy = |z| (\cos[\phi] - i \sin[\phi]) = r e^{-i\phi}$$

$$x = \operatorname{Re}[z]$$

$$y = \operatorname{Im}[z]$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\phi = \arg[z] = \operatorname{atan2}(y, x)$$

$$a = e^{\ln[a]}$$

■ Power series definition

complex z

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

■ Limit definition

complex z

$$e^z = \lim_{n \rightarrow \infty} \left(1 + \frac{z}{n} \right)^n$$

■ Derivative

■ Using Power series definition

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos[x] = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\sin[x] = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\begin{aligned}
e^{iz} &= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \frac{(iz)^6}{6!} + \frac{(iz)^7}{7!} + \frac{(iz)^8}{8!} + \dots \\
&= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} - \frac{z^6}{6!} - \frac{iz^7}{7!} + \frac{z^8}{8!} + \dots \\
&= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} + \dots\right) + i \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right) \\
&= \cos[z] + i \sin[z]
\end{aligned}$$

■ Using calculus

$$\begin{aligned}
f[x] &= e^{-ix} (\cos[x] + i \sin[x]) \\
\frac{d}{dx} f[x] &= -ie^{-ix} (\cos[x] + i \sin[x]) + e^{-ix} (-\sin[x] + i \cos[x]) \\
&= e^{-ix} (-i \cos[x] + \sin[x] - \sin[x] + i \cos[x]) \\
&= 0 \\
\therefore f[x] &= C \\
f[0] &= 1 \\
\therefore C &= 1 \\
e^{-ix} (\cos[x] + i \sin[x]) &= 1 \\
e^{ix} &= (\cos[x] + i \sin[x])
\end{aligned}$$

■ differential equations

$$\begin{aligned}
z &= \cos[x] + i \sin[x] \\
\frac{dz}{dx} &= -\sin[x] + i \cos[x] \\
i^2 &= -1 \\
\frac{dz}{dx} &= i^2 \sin[x] + i \cos[x] = i (\cos[x] + i \sin[x]) = iz \\
\frac{1}{z} \frac{dz}{dx} &= i \\
\int \frac{1}{z} dz &= \int i dx \\
\ln[z] &= ix + C \\
\ln[z] &= C \\
z = \cos[x] + i \sin[x] &= \cos[0] + i \sin[0] = 1 \\
\ln[1] &= C \\
C &= 0 \\
\ln[z] &= ix \\
z &= e^{ix} \\
e^{ix} &= \cos[x] + i \sin[x]
\end{aligned}$$

■ cis

$$\text{cis}[\theta] = e^{\mathbf{i}\theta} = \text{Cos}[\theta] + \mathbf{i} \text{Sin}[\theta]$$